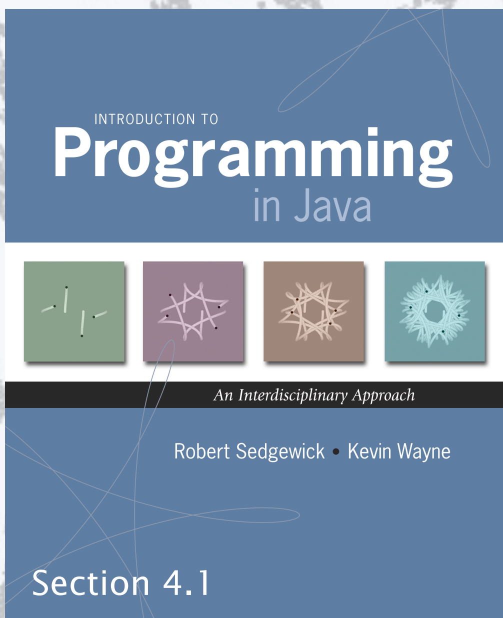

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8. Performance Analysis

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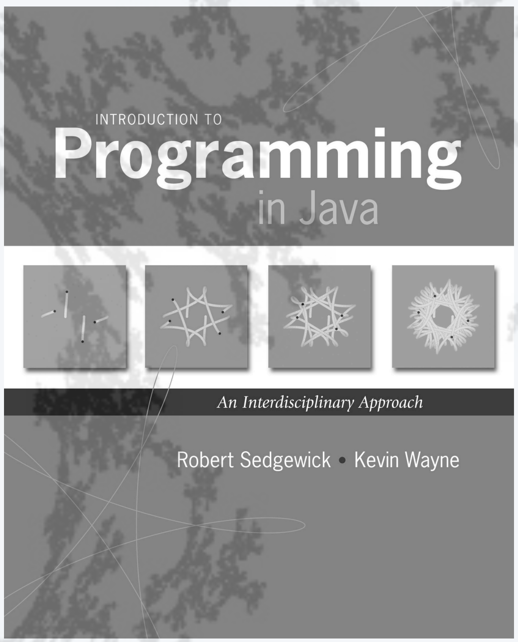

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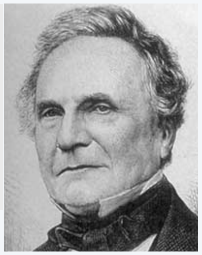
8. Performance analysis

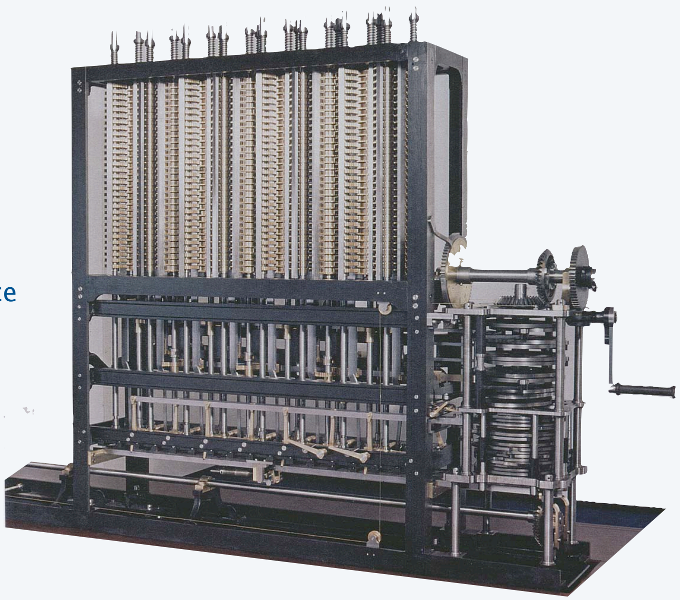
- **The challenge**
- Empirical analysis
- Mathematical models
- Meeting the challenge
- Familiar examples

The challenge (since the earliest days of computing machines)

" As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise—By what course of calculation can these results be arrived at by the machine in the shortest time?"

– Charles Babbage





Difference Engine #2

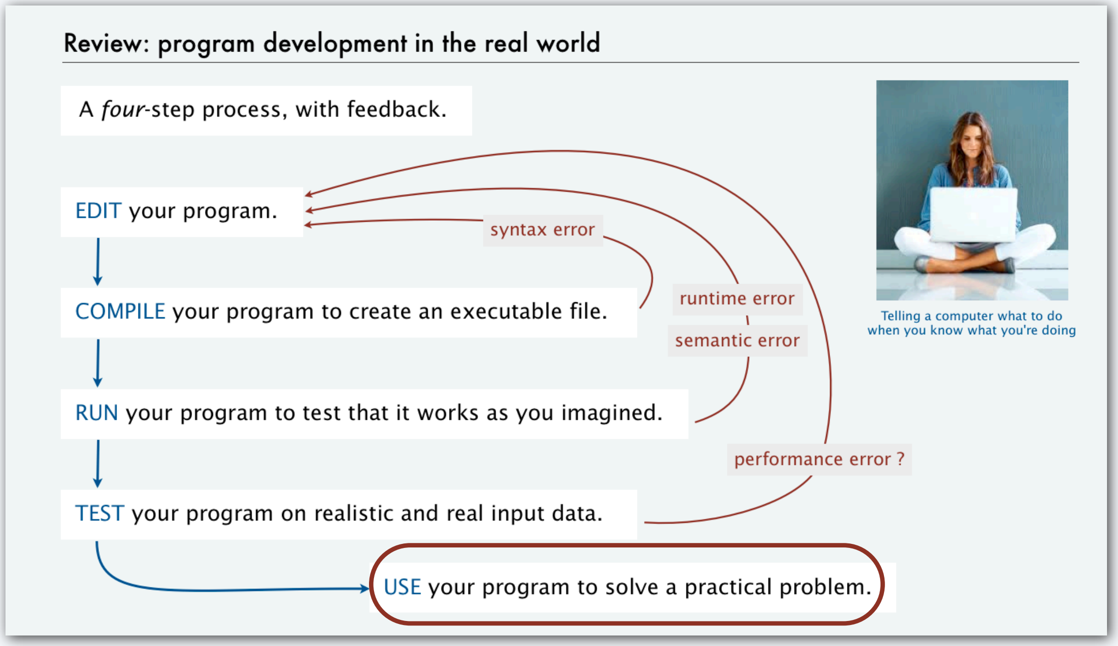
Designed by Charles Babbage, c. 1848

Built by London Science Museum, 1991

Q. How many times do you have to turn the crank?

The challenge (modern version)

Q. Will I be able to use my program to solve a large practical problem?



Q. If not, how might I understand its performance characteristics so as to improve it?

Key insight (Knuth 1970s). Use the *scientific method* to understand performance.

Three reasons to study program performance

1. To predict program behavior
   - Will my program finish?
   - When will my program finish?

```
public class HelloWorld
{
    public static void main(String[] args)
    {
        System.out.println("Hello, World");
    }
}
```

2. To compare algorithms and implementations.
   - Will this change make my program faster?
   - How can I make my program faster?

3. To develop a basis for understanding the problem and for designing new algorithms
   - Enables new technology.
   - Enables new research.

An *algorithm* is a method for solving a problem that is suitable for implementation as a computer program.

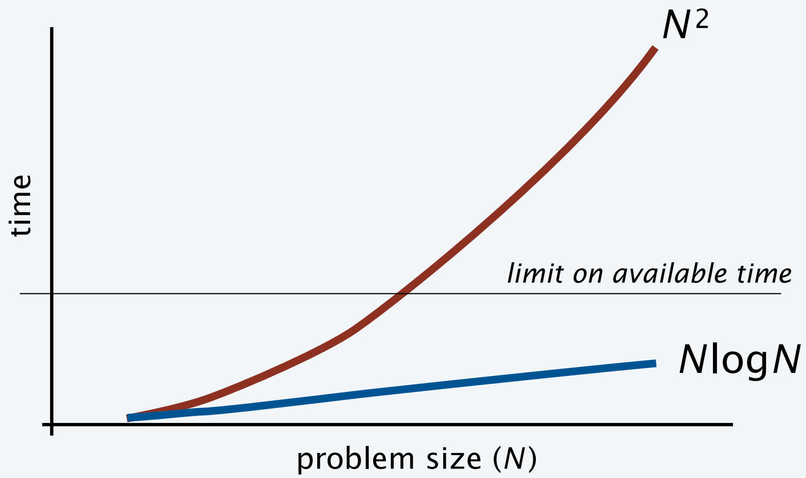
We study several algorithms later in this course.
Taking more CS courses?
You'll learn dozens of algorithms.

An algorithm design success story

N-body simulation

- Goal: Simulate gravitational interactions among N bodies.
- Brute-force algorithm requires N^2 steps.
- Issue (1970s): Too slow to address scientific problems of interest.
- Success story: *Barnes-Hut* algorithm uses $N \log N$ steps and *enables new research*.

Andrew Appel
PU '81
senior thesis

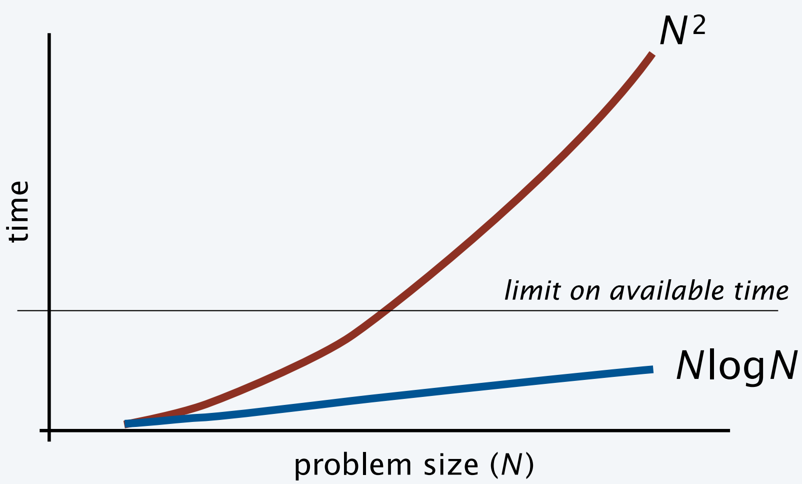


Another algorithm design success story

Fast Fourier transform

- Goal: Break down waveform of N samples into periodic components.
- Applications: digital signal processing, spectroscopy, ...
- Brute-force algorithm requires N^2 steps.
- Issue (1950s): Too slow to address commercial applications of interest.
- Success story: *FFT* algorithm requires $N \log N$ steps and *enables new technology*.

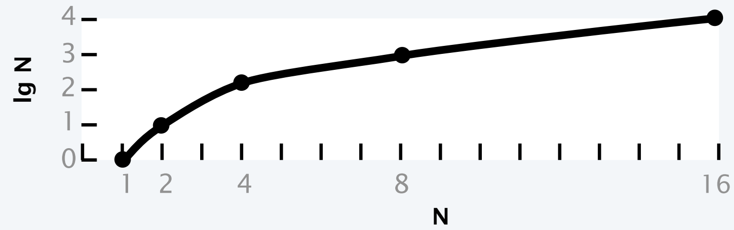
John Tukey
1915–2000



Quick aside: binary logarithms

Def. The *binary logarithm* of a number N (written lg N) is the number x satisfying $2^x = N$.

or $\log_2 N$



Q. How many recursive calls for convert(N)?

```
public static String convert(int N)
{
   if (N == 1) return "1";
   return convert(N/2) + (N % 2);
}
```

Frequently encountered values

N	*approximate value*	lg$N$	$\log_{10}N$
2^{10}	1 thousand	10	3.01
2^{20}	1 million	20	6.02
2^{30}	1 billion	30	9.03

A. Largest integer less than lg N (written $\lfloor \lg N \rfloor$). ← Prove by induction. Details in "sorting and searching" lecture.

Fact. The number of bits in the binary representation of N is $1 + \lfloor \lg N \rfloor$.

Fact. Binary logarithms arise in the study of algorithms based on recursively solving problems half the size (*divide-and-conquer algorithms*), like convert, FFT and Barnes-Hut.

An algorithmic challenge: 3-sum problem

Three-sum. Given *N* integers, enumerate the triples that sum to 0.

For starters, just count them (might choose to process them all).

```
public class ThreeSum
{
   public static int count(int[] a)
   {  // See next slide. }
   public static void main(String[] args)
   {
      int[] a = StdIn.readAllInts();
      StdOut.println(count(a));
   }
}
```

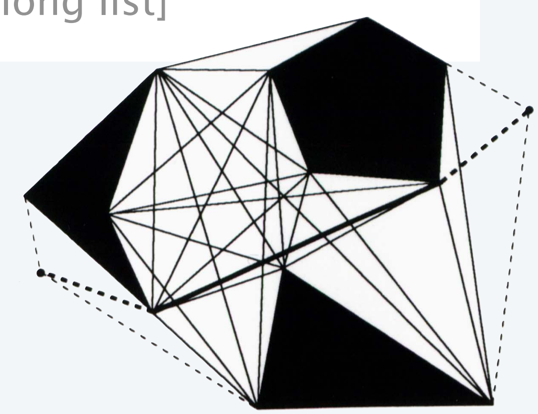
```
% more 6ints.txt
30 -30 -20 -10 40  0

% java ThreeSum <  6ints.txt
3
```

30	-30	0
30	-20	-10
-30	-10	40

Applications in computational geometry

- Find collinear points.
- Does one polygon fit inside another?
- Robot motion planning.
- [a surprisingly long list]



Q. Can we solve this problem for $N = 1$ million?

Three-sum implementation

"Brute force" starting point

- Process all possible triples.
- Increment counter when sum is 0.

i	0	1	2	3	4	5
a[i]	30	-30	-20	-10	40	0

```
public static int count(int[] a)
{
   int N = a.length;
   int cnt = 0;
   for (int i = 0; i < N; i++)
      for (int j = i+1; j < N; j++)
         for (int k = j+1; k < N; k++)
            if (a[i] + a[j] + a[k] == 0)
               cnt++;
   return cnt;
}
```

Keep i < j < k to avoid processing each triple 6 times

$\binom{N}{3}$ triples with i < j < k

i	j	k	a[i]	a[j]	a[k]
0	1	2	30	-30	-20
		3	30	-30	-10
		4	30	-30	40
		5	30	-30	0
	2	3	30	-20	-10
		4	30	-20	40
		5	30	-20	0
	3	4	30	-10	40
		5	30	-10	0
	4	5	30	40	0
1	2	3	-30	-20	-10
		4	-30	-20	40
		5	-30	-20	0
	3	4	-30	-10	40
		5	-30	-10	0
	4	5	-30	40	0
2	3	4	-20	-10	40
		5	-20	-10	0
	4	5	-20	40	0
3	4	5	-10	40	0

Q. How much time will this program take for $N = 1$ million?

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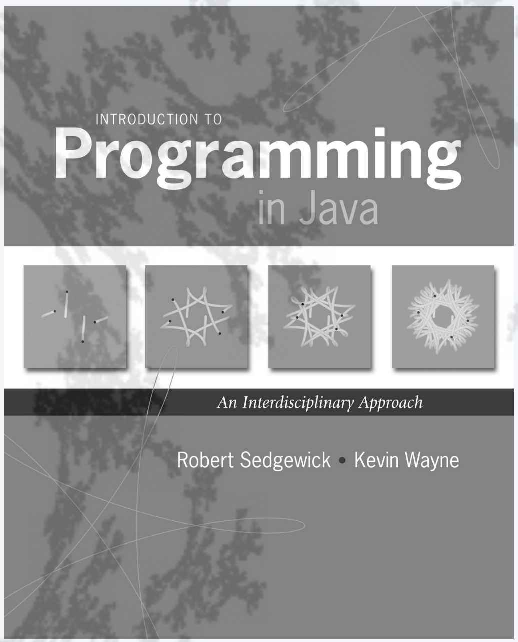

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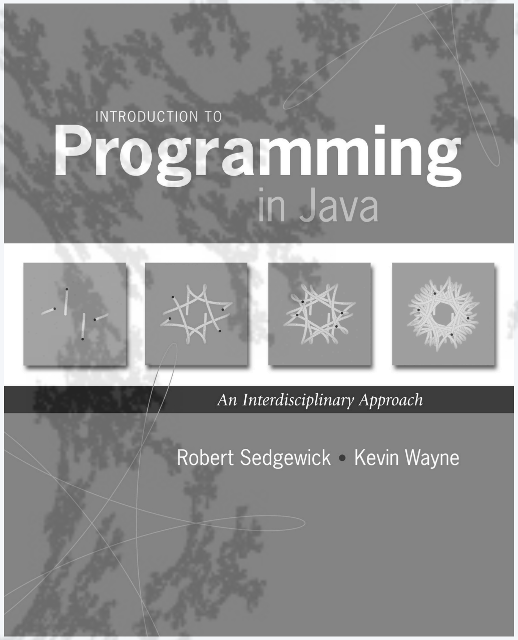

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A first step in analyzing running time

Find representative inputs

- Option 1: Collect actual potential input data.
- Option 2: Write a program to generate representative inputs.

Input generator for ThreeSum

```
public class Generator
{  // Generate N integers in [-M, M)
   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      int M  = Integer.parseInt(args[1]);
      for (int i = 0; i < N; i++)
         StdOut.println(StdRandom.uniform(-M, M));
   }
}
```

```
% java Generator 10 1000000
28773
-807569
-425582
594752
600579
-483784
-861312
-690436
-732636
360294
```

not much chance of a 3-sum

```
% java Generator 10 10
-2
1
-4
1
-2
-10
-4
1
0
-7
```

good chance of a 3-sum

Empirical analysis

Run experiments

- Start with a moderate size.
- Measure and record running time.
- Double size.
- Repeat.
- Tabulate and plot results.

Measure running time

Replace println() in ThreeSum with this code.

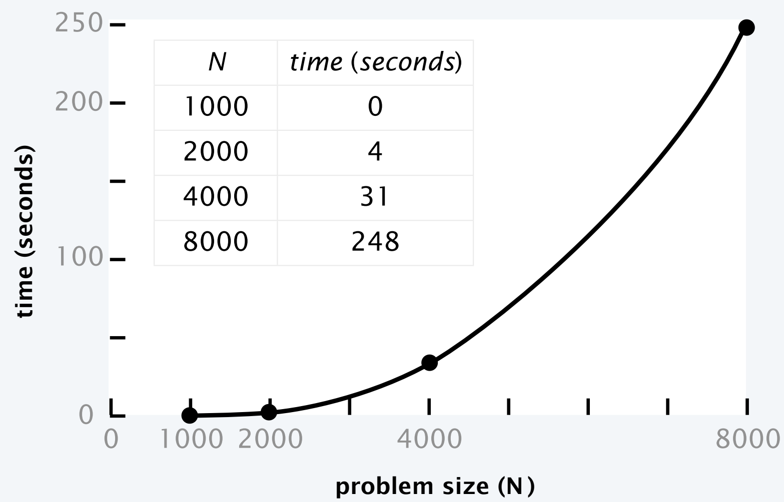
```
double start = System.currentTimeMillis()/1000.0;
int cnt = count(a);
double now   = System.currentTimeMillis()/1000.0;
int time = Math.round(now - start);
StdOut.println(cnt + " (" + time + " seconds)");
```

Run experiments

```
% java Generator 1000 1000000 | java ThreeSum
59 (0 seconds)
% java Generator 2000 1000000 | java ThreeSum
522 (4 seconds)
% java Generator 4000 1000000 | java ThreeSum
3992 (31 seconds)
% java Generator 8000 1000000 | java ThreeSum
31903 (248 seconds)
```

Tabulate and plot results

N	*time (seconds)*
1000	0
2000	4
4000	31
8000	248



Aside: experimentation in CS

is *virtually free*, particularly by comparison with other sciences.

Chemistry

Biology

Computer Science

Physics

Bottom line. *No excuse* for not running experiments to understand costs.

Data analysis

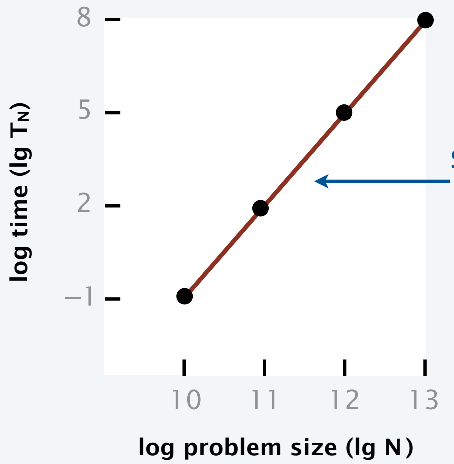
Curve fitting

- Plot on *log-log scale.*
- If points are on a straight line (often the case), a *power law* holds—a curve of the form aN^b fits.
- The exponent b is the slope of the line.
- Solve for a with the data.

N	T_N	$\lg N$	$\lg T_N$	$4.84 \times 10^{-10} \times N^3$
1000	0.5	10	−1	0.5
2000	4	11	2	4
4000	31	12	5	31
8000	248	13	8	248

✓

log-log plot



Do the math

x-intercept (use lg in anticipation of next step)

$$\lg T_N = \lg a + 3 \lg N$$ equation for straight line of slope 3

$$T_N = aN^3$$ raise 2 to a power of both sides

$$248 = a \times 8000^3$$ substitute values from experiment

$$a = 4.84 \times 10^{-10}$$ solve for a

$$T_N = 4.84 \times 10^{-10} \times N^3$$ substitute

a curve that fits the data

Prediction and verification

Hypothesis. Running time of ThreeSum is $4.84 \times 10^{-10} \times N^3$.

Prediction. Running time for $N = 16{,}000$ will be 1982 seconds.

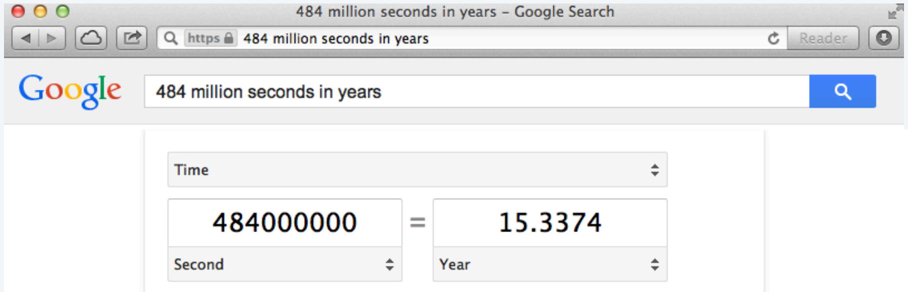
about half an hour

```
% java Generator 16000 1000000 | java ThreeSum
31903 (1985 seconds)
```

✓

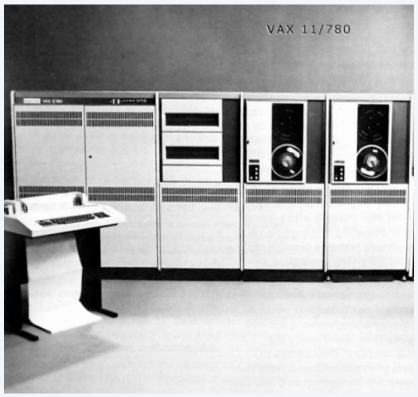
Q. How much time will this program take for $N = 1$ million?

A. 484 million seconds (more than 15 years).



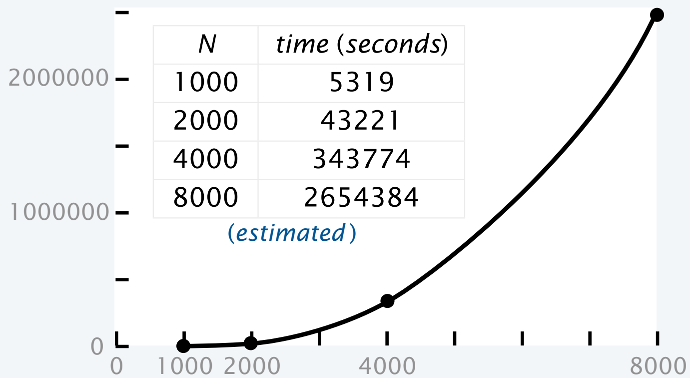
Another hypothesis

1970s



VAX 11/780

$5.2 \times 10^{-6} \times N^3$ seconds



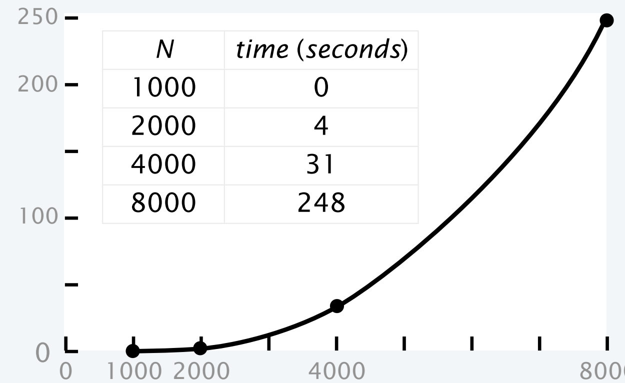
N	*time (seconds)*
1000	5319
2000	43221
4000	343774
8000	2654384

(estimated)

2010s: 10,000+ times faster

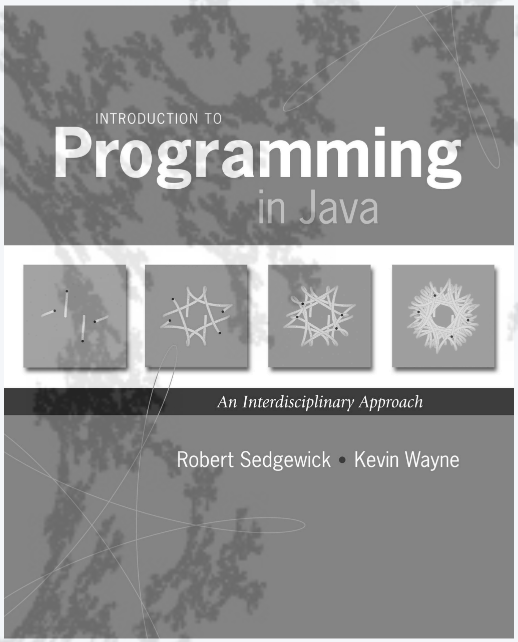
Macbook Air

$4.84 \times 10^{-10} \times N^3$ seconds



N	*time (seconds)*
1000	0
2000	4
4000	31
8000	248

Hypothesis. Running times on different computers only differ by a constant factor.



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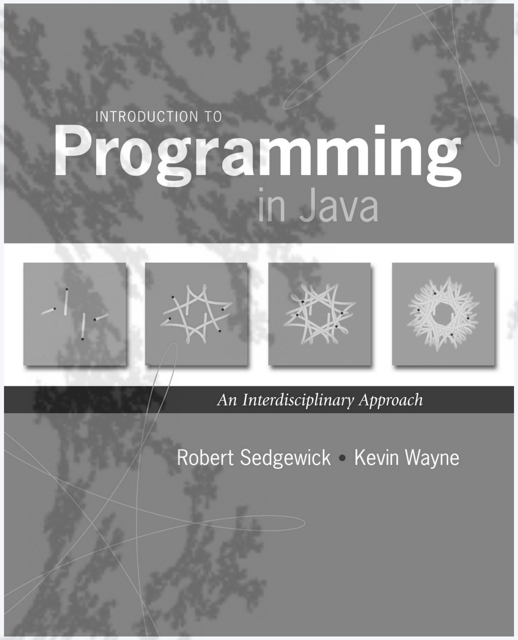

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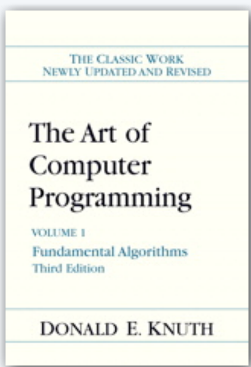
Mathematical models for running time

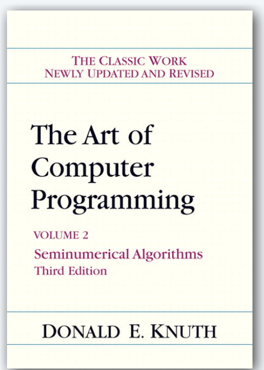
Q. Can we write down an accurate formula for the running time of a computer program?

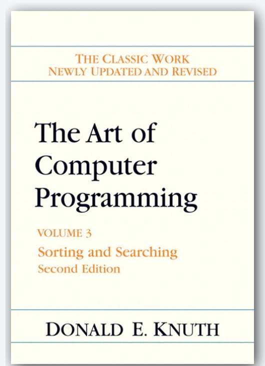
A. (Prevailing wisdom, 1960s) No, too complicated.

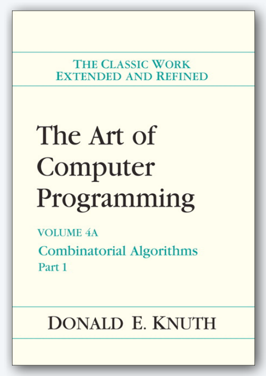
A. (D. E. Knuth, 1968–*present*) Yes!

- Determine the set of operations.
- Find the *cost* of each operation (depends on computer and system software).
- Find the *frequency of execution* of each operation (depends on algorithm and inputs).
- Total running time: sum of cost × frequency for all operations.









Don Knuth
1974 Turing Award

Warmup: 1-sum

```
public static int count(int[] a)
{
   int N = a.length;
   int cnt = 0;
   for (int i = 0; i < N; i++)
      if (a[i] == 0)
         cnt++;
   return cnt;
}
```

Note that frequency of increments depends on input.

operation	*cost*	*frequency*
function call/return	20 *ns*	1
variable declaration	2 *ns*	2
assignment	1 *ns*	2
less than compare	1/2 *ns*	$N + 1$
equal to compare	1/2 *ns*	N
array access	1/2 *ns*	N
increment	1/2 *ns*	between N and $2N$

representative estimates (with some poetic license); knowing exact values may require study and experimentation.

Q. Formula for total running time ?

A. $cN + 26.5$ nanoseconds, where c is between 2 and 2.5, depending on input.

Warmup: 2-sum

```
public static int count(int[] a)
{
   int N = a.length;
   int cnt = 0;
   for (int i = 0; i < N; i++)
      for (int j = i+1; j < N; j++)
         if (a[i] + a[j] == 0)
            cnt++;
   return cnt;
}
```

operation	*cost*	*frequency*
function call/return	20 *ns*	1
variable declaration	2 *ns*	$N + 2$
assignment	1 *ns*	$N + 2$
less than compare	1/2 *ns*	$(N + 1)(N + 2)/2$
equal to compare	1/2 *ns*	$N(N - 1)/2$
array access	1/2 *ns*	$N(N - 1)$
increment	1/2 *ns*	between $N(N - 1)/2$ and $N(N - 1)$

exact counts tedious to derive

$$\# \; i < j \; = \binom{N}{2} = \frac{N(N-1)}{2}$$

Q. Formula for total running time ?

A. $c_1 N^2 + c_2 N + c_3$ nanoseconds, where... [complicated definitions].

Simplifying the calculations

Tilde notation

- Use only the fastest-growing term.
- Ignore the slower-growing terms.

Rationale

- When N is large, ignored terms are negligible.
- When N is small, *everything* is negligible.

Def. $f(N) \sim g(N)$ means $f(N)/g(N) \to 1$ as $N \to \infty$

Example. $6N^2 + 20N + 5 \sim 6N^2$

6,020,005 for $N = 1{,}000$

6,000,000 for $N = 1{,}000$, within .3%

Q. Formula for 2-sum running time when count is not large (typical case)?

eliminate dependence on input

A. $\sim 6N^2$ nanoseconds.

Mathematical model for 3-sum

```
public static int count(int[] a)
{
   int N = a.length;
   int cnt = 0;
   for (int i = 0; i < N; i++)
      for (int j = i+1; j < N; j++)
         for (int k = j+1; k < N; k++)
            if (a[i] + a[j] + a[k] == 0)
               cnt++;
   return cnt;
}
```

operation	*cost*	*frequency*
function call/return	20 *ns*	1
variable declaration	2 *ns*	$\sim N$
assignment	1 *ns*	$\sim N$
less than compare	1/2 *ns*	$\sim N^3/6$
equal to compare	1/2 *ns*	$\sim N^3/6$
array access	1/2 *ns*	$\sim N^3/2$
increment	1/2 *ns*	$\sim N^3/6$

$$\# \; i < j < k = \binom{N}{3} = \frac{N(N-1)(N-2)}{6} \sim \frac{N^3}{6}$$

Q. Formula for total running time when return value is not large (typical case)?

A. $\sim N^3/2$ nanoseconds. ✓ ⟵ matches $4.84 \times 10^{-10} \times N^3$ empirical hypothesis

Context

Scientific method

- *Observe* some feature of the natural world.
- *Hypothesize* a model consistent with observations.
- *Predict* events using the hypothesis.
- *Verify* the predictions by making further observations.
- *Validate* by refining until hypothesis and observations agree.

Francis Bacon
1561–1626

René Descartes
1596–1650

John Stuart Mill
1806–1873

Empirical analysis of programs

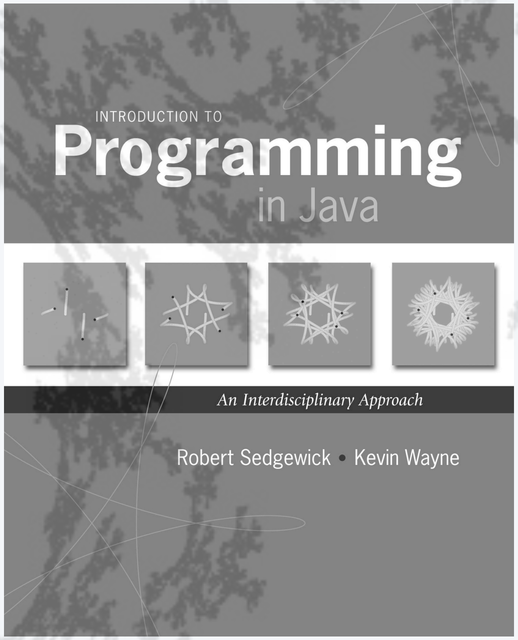
- "Feature of natural world" is time taken by a program on a computer.
- Fit a curve to experimental data to get a formula for running time as a function of N.
- Useful for predicting, but not *explaining*.

Mathematical analysis of algorithms

- Analyze *algorithm* to develop a formula for running time as a function of N.
- Useful for predicting *and* explaining.
- Might involve advanced mathematics.
- Applies to any computer.

Good news. Mathematical models are easier to formulate in CS than in other sciences.

COMPUTER SCIENCE
SEDGEWICK / WAYNE



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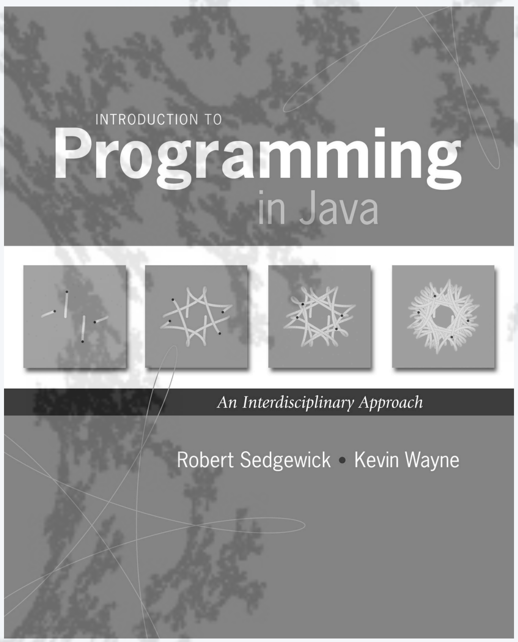

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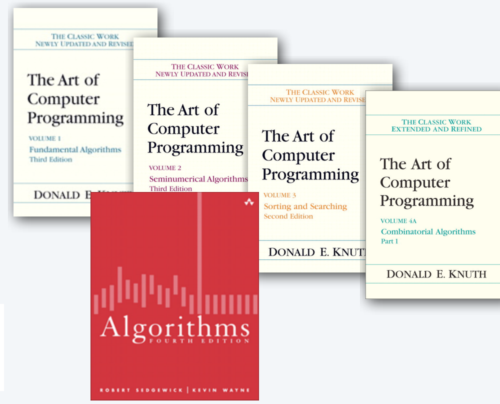
Key questions and answers

Q. Is the running time of my program ~ $a\,N^b$ seconds?

A. Yes, there's good chance of that. Might also have a $(\lg N)^c$ factor.

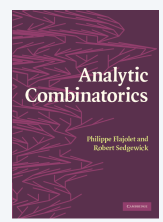
Q. How do you know?



A. Computer scientists have applied such models for decades to many, many specific algorithms and applications.

A. Programs are built from simple constructs (examples to follow).

A. Real-world data is also often simply structured.



A. Deep connections exist between such models and a wide variety of discrete structures (including some programs).

Doubling method

Hypothesis. The running time of my program is $T_N \sim a\,N^b$.

Consequence. As N increases, $T_N / T_{N/2}$ approaches 2^b.

no need to calculate a (!)

Proof: $$\frac{a(2N)^b}{aN^b} = 2^b$$

Doubling method

- Start with a moderate size.
- Measure and record running time.
- Double size.
- Repeat while you can afford it.
- Verify that *ratios* of running times approach 2^b.
- Predict by *extrapolation*:

  multiply by 2^b to estimate T_{2N} and repeat.

3-sum example

N	T_N	$T_N / T_{N/2}$
1000	0.5	
2000	4	8
4000	31	7.75
8000	248	8
16000	$248 \times 8 = 1984$	8
32000	$248 \times 8^2 = 15872$	8
...		
1024000	$248 \times 8^7 = 520093696$	8

✓

math model says running time should be aN^3
$2^3 = 8$

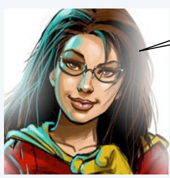
Bottom line. It is often *easy* to meet the challenge of predicting performance.

Caveats

It is *sometimes* not so easy to meet the challenge of predicting performance.

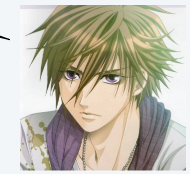
There are many other apps running on my computer!

Your *input* model is too simple: My real input data is completely different.

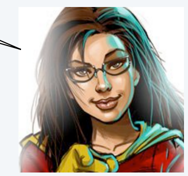


Your *machine* model is too simple: My computer has parallel processors and a cache.

We need more terms in the math model: $N \lg N + 100N$?



What happens when the leading term oscillates?



Where's the log factor?

$$\frac{a(2N)^b(\lg(2N))^c}{aN^b(\lg N)^c} = 2^b\left(1 + \frac{1}{(\lg N)}\right)^c \sim 2^b$$

Good news. Doubling method is *robust* in the face of many of these challenges.

Order of growth

Def. If a function $f(N) \sim ag(N)$ we say that $g(N)$ is the *order of growth* of the function.

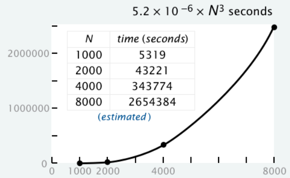
Hypothesis. Order of growth is a property of the *algorithm*, not the computer or the system.

Experimental validation

Another hypothesis

1970s


VAX 11/780

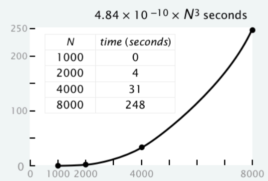


N	*time (seconds)*
1000	5319
2000	43221
4000	343774
8000	2654384

(estimated)

2010s: 10,000+ times faster


Macbook Air



N	*time (seconds)*
1000	0
2000	4
4000	31
8000	248

Hypothesis. Running times on different computers only differ by a constant factor.

When we move a program to a computer that is X times faster, we expect the program to be X times faster.

Explanation with mathematical model

Mathematical model for 3-sum

```
public static int count(int[] a)
{
   int N = a.length;
   int cnt = 0;
   for (int i = 0; i < N; i++)
      for (int j = i+1; j < N; j++)
         for (int k = j+1; k < N; k++)
            if (a[i] + a[j] + a[k] == 0)
               cnt++;
   return cnt;
}
```

operation	*cost*	*frequency*
function call/return	20 *ns*	1
variable declaration	2 *ns*	$\sim N$
assignment	1 *ns*	$\sim N$
less than compare	1/2 *ns*	$\sim N^3/6$
equal to compare	1/2 *ns*	$\sim N^3/6$
array access	1/2 *ns*	$\sim N^3/2$
increment	1/2 *ns*	$\sim N^3/6$

$$\# \; i < j < k = \binom{N}{3} = \frac{N(N-1)(N-2)}{6} \sim \frac{N^3}{6}$$

Q. Formula for total running time when return value is not large (typical case)?

A. $\sim N^3/2$ nanoseconds. ✓ ⟵ matches $4.84 \times 10^{-10} \times N^3$ empirical hypothesis

Machine- and system-dependent features of the model are all small constants.

Order of growth

Hypothesis. The order of growth of the running time of my program is $N^b (\lg N)^c$.

Evidence. Known to be true for many, many programs with simple and similar structure.

Linear (N)

```
for (int i = 0; i < N; i++)
  ...
```

Quadratic (N²)

```
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
     ...
```

Cubic (N³)

```
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      for (int k = j+1; k < N; k++)
        ...
```

Logarithmic (lg N)

```
public static void f(int N)
{
   if (N == 0) return;
   ... f(N/2)...
}
```

Linearithmic (Nlg N)

```
public static void f(int N)
{
   if (N == 0) return;
   ... f(N/2)...
   ... f(N/2)...
}
```

Exponential (2^N)

```
public static void f(int N)
{
   if (N == 0) return;
   ... f(N-1)...
   ... f(N-1)...
}
```

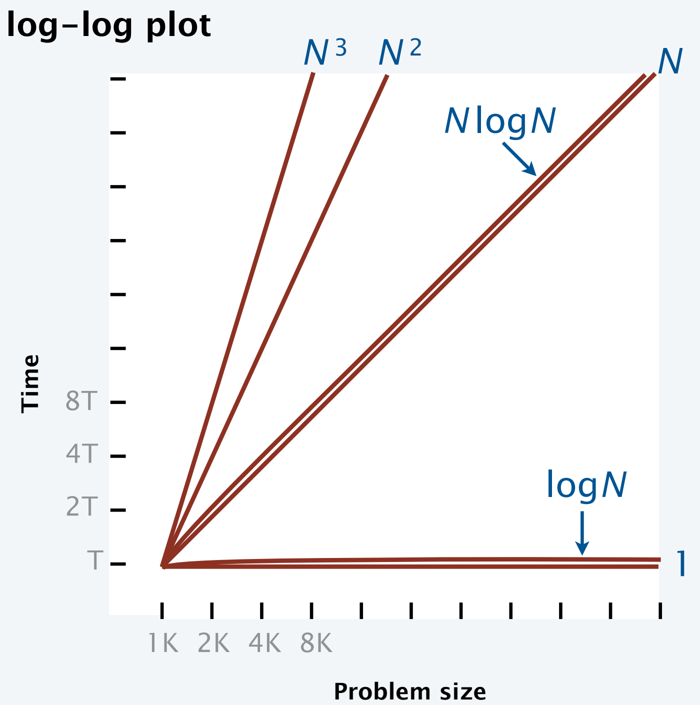
ignore for practical purposes
(any significant growth is infeasible)

Stay tuned for specific examples.

Order of growth classifications

order of growth		slope of line in log-log plot (b)	factor for doubling method (2^b)
description	*function*		
constant	1	0	1
logarithmic	$\log N$	0	1
linear	N	1	2
linearithmic	$N \log N$	1	2
quadratic	N^2	2	4
cubic	N^3	3	8

if input size doubles
running time increases
by this factor



math model may
have log factor

If math model gives order of growth, use doubling method to validate 2^b ratio.

If not, use doubling method and solve for $b = \lg(T_N / T_{N/2})$ to estimate order of growth to be N^b.

An important implication

Moore's Law. Computer power increases by a factor of 2 every 2 years.

Q. My *problem size* also doubles every 2 years. How much do I need to spend to get my job done?

a very common situation: weather prediction, transaction processing, cryptography...

Do the math

$T_N = aN^3$ running time today

$T_{2N} = (a/2)(2N)^3$ running time in 2 years

$= 4aN^3$

$= 4T_N$

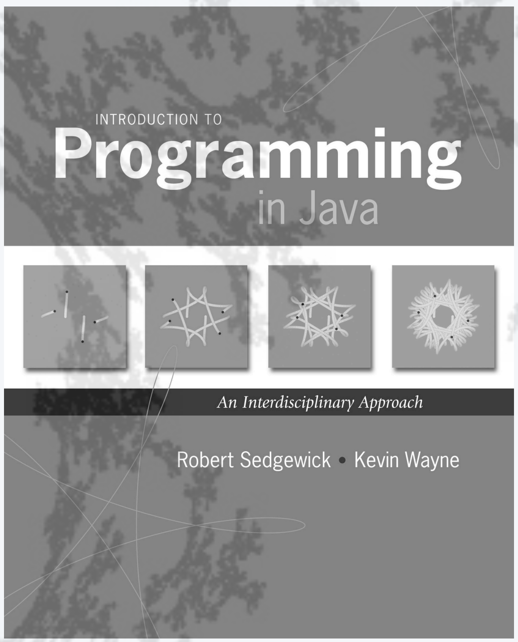
	now	*2 years from now*	*4 years from now*		*2M years from now*
N	\$X	\$X	\$X	...	\$X
$N \log N$	\$X	\$X	\$X	...	\$X
N^2	\$X	\$2X	\$4X	...	$\$2^M$ X
N^3	\$X	\$4X	\$16X	...	$\$4^M$ X

A. You can't afford to use a quadratic algorithm (or worse) to address increasing problem sizes.

Meeting the challenge

Doubling experiments provide good insight on program performance

- Best practice to plan realistic experiments for debugging, anyway.
- Having *some* idea about performance is better than having *no* idea.
- *Performance matters* in many, many situations.



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8. Performance analysis

- The challenge
- Empirical analysis
- Mathematical models
- **Meeting the challenge**
- Familiar examples

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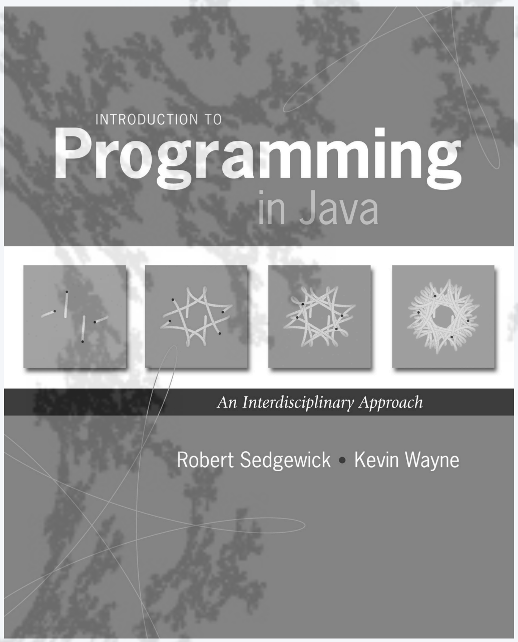

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Example: Gambler's ruin simulation

Q. How long to compute chance of doubling 1 million dollars?

```
public class Gambler
{
    public static void main(String[] args)
    {
      int stake  = Integer.parseInt(args[0]);
      int goal   = Integer.parseInt(args[1]);
      int trials = Integer.parseInt(args[2]);
double start = System.currentTimeMillis()/1000.0;
      int wins   = 0;
      for (int i = 0; i < trials; i++)
      {
         int t = stake;
         while (t > 0 && t < goal)
         {
            if (Math.random() < 0.5) t++;
            else                     t--;
         }
         if (t == goal) wins++;
      }
double now = System.currentTimeMillis()/1000.0;
long time = Math.round(now - start);
      StdOut.print(wins + " wins of " + trials);
StdOut.println(" (" + time  +  " seconds)");
    }
}
```

```
% java Gambler 1000 2000 100
53 wins of 100 (4 seconds)
% java Gambler 2000 4000 100
52 wins of 100 (17 seconds)
% java Gambler 4000 8000 100
55 wins of 100 (56 seconds)
% java Gambler 8000 16000 100
53 wins of 100 (286 seconds)
```

```
% java Gambler 16000 32000 100
48 wins of 100 (1172 seconds)
```

N	T_N	$T_N / T_{N/2}$
1000	4	
2000	17	4.25
4000	56	3.29
8000	286	5.10
16000	1172	4.09 ✓
32000	$1172 \times 4 = 4688$	4
...		
1024000	$1172 \times 4^6 = 4800512$	4

math model says order of growth should be N^2

A. 4.8 million seconds (about 2 months).

Another example: Factoring

Q. How large a number can I factor in a day (86,400 seconds)?

```
public class Factors
{
   public static void main(String[] args)
   {
double start = System.currentTimeMillis()/1000.0;
      long N = Long.parseLong(args[0]);
      for ( long i = 2; i <= N/i; i++)
      {
         while (N % i == 0)
         {
            System.out.print(i + " ");
            N = N / i;
         }
      }
double now = System.currentTimeMillis()/1000.0;
long time = Math.round(now - start);
      if (N > 1) System.out.print(N);
      else       System.out.print();
StdOut.println(" (" + time  +  " seconds)");
   }
}
```

A. About 2^{84} (26 digits).

N	T_N	$T_N/T_{N/4}$
$2^{58} + 69$	12	
$2^{59} + 131$	18	
$2^{60} + 33$	25	2.08
$2^{61} + 15$	35	1.94
$2^{62} + 135$	49	1.96 ✓
...		2
2^{84} + ...	$49 \times 2^{11} = 100352$	2

```
% java Factors 288230376151711813
288230376151711813 (12 seconds)
% java Factors 576460752303423619
576460752303423619 (18 seconds)
% java Factors 1152921504606847009
1152921504606847009 (25 seconds)
% java Factors 2305843009213693967
2305843009213693967 (35 seconds)
% java Factors 4611686018427388039
4611686018427388039 (49 seconds)
```

Tricky: math model says order of growth should be $\sqrt{N}$

$$\frac{a(4N)^b}{aN^b} = 2$$

$$4^b = 2$$

$$b = 1/2$$

TEQ on performance

Q. Let T_N be the running time of program Mystery and consider these experiemnts:

```
public static Mystery
{
   ...
   int N = Integer.parseInt(args[0]);
   ...
}
```

N	T_N (in seconds)	$T_N / T_{N/2}$
1000	5	
2000	20	4
4000	80	4
8000	320	4

Q. Predict the running time for $N = 64{,}000$.

Q. Estimate the order of growth.

TEQ on performance

Q. Let T_N be the running time of program Mystery and consider these experiemnts.

```
public static Mystery
{
   ...
   int N = Integer.parseInt(args[0]);
   ...
}
```

N	T_N (in seconds)	$T_N / T_{N/2}$
1000	5	
2000	20	4
4000	80	4
8000	320	4
16000	$320 \times 4 = 1280$	4
32000	$1280 \times 4 = 5120$	4
64000	$5120 \times 4 = 20480$	4

Q. Predict the running time for $N = 64{,}000$.

A. 20480 seconds.

Q. Estimate the order of growth.

A. N^2, since $\lg 4 = 2$.

Another example: Coupon collector

Q. How long to simulate collecting 1 million coupons?

```
public class Collector
{
    public static void main(String[] args)
    {
        int N = Integer.parseInt(args[0]);
        int trials = Integer.parseInt(args[1]);
        int cardcnt = 0;
         boolean[] found;
double start = System.currentTimeMillis()/1000.0;

        for (int i = 0; i < trials; i++)
         {
            int valcnt = 0;
           found = new boolean[N];
            while (valcnt < N)
            {
              int val = (int) (Math.random() * N);
               cardcnt++;
              if (!found[val])
                  { valcnt++; found[val] = true; }
            }
         }
double now = System.currentTimeMillis()/1000.0;
long time = Math.round(now - start);
         System.out.print(N + " " + (N*Math.log(N) + .57721*N) + " ");
         System.out.print(cardcnt/trials);
StdOut.println(" (" + time  +  " seconds)");
    }
}
```

N	T_N	$T_N/T_{N/2}$
125000	7	
250000	14	2
500000	31	2.21 ✓
1000000	$31 \times 2 = 63$	2

math model says order of growth should be $N \log N$

```
% java Collector 125000 100
125000 1539159.8770355547 1518646 (7 seconds)
% java Collector 250000 100
250000 3251606.5492110956 3173727 (14 seconds)
% java Collector 500000 100
500000 6849786.688702164 6772679 (31 seconds)
```

```
% java Collector 1000000 100
1000000 1.4392720557964273E7 14368813 (66 seconds)
```
✓

A. About 1 minute. ← might run out of memory trying for 1 billion

Analyzing typical memory requirements

A *bit* is 0 or 1 and the basic unit of memory.

A *byte* is eight bits and the smallest addressable unit.

1 *megabyte* (MB) is 1 million bytes.

1 *gigabyte* (GB) is 1 billion bytes.

Primitive-type values

type	*bytes*	
`boolean`	1	← Note: *not* 1 bit
`char`	1	
`int`	4	
`float`	4	
`long`	8	
`double`	8	

System-supported data structures

type	*bytes*
`int[N]`	$4N + 16$
`double[N]`	$8N + 16$
`int[N][N]`	$4N^2 + 20N + 16 \sim 4N^2$
`double[N][N]`	$8N^2 + 20N + 16 \sim 8N^2$
`String`	$2N + 40$

Example. 2000-by-2000 `double` array uses ~32MB.

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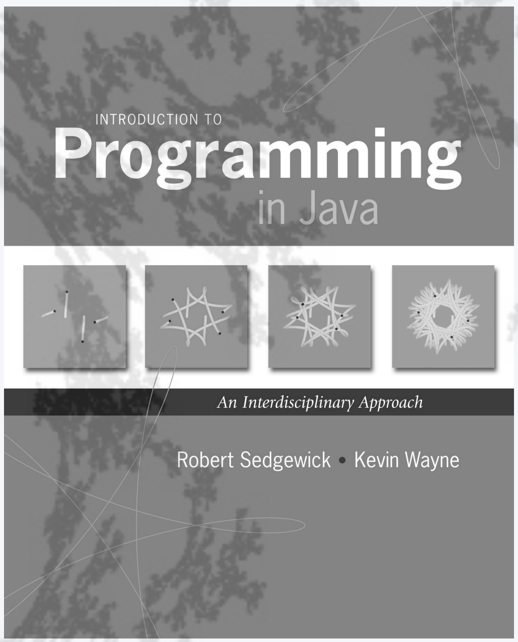

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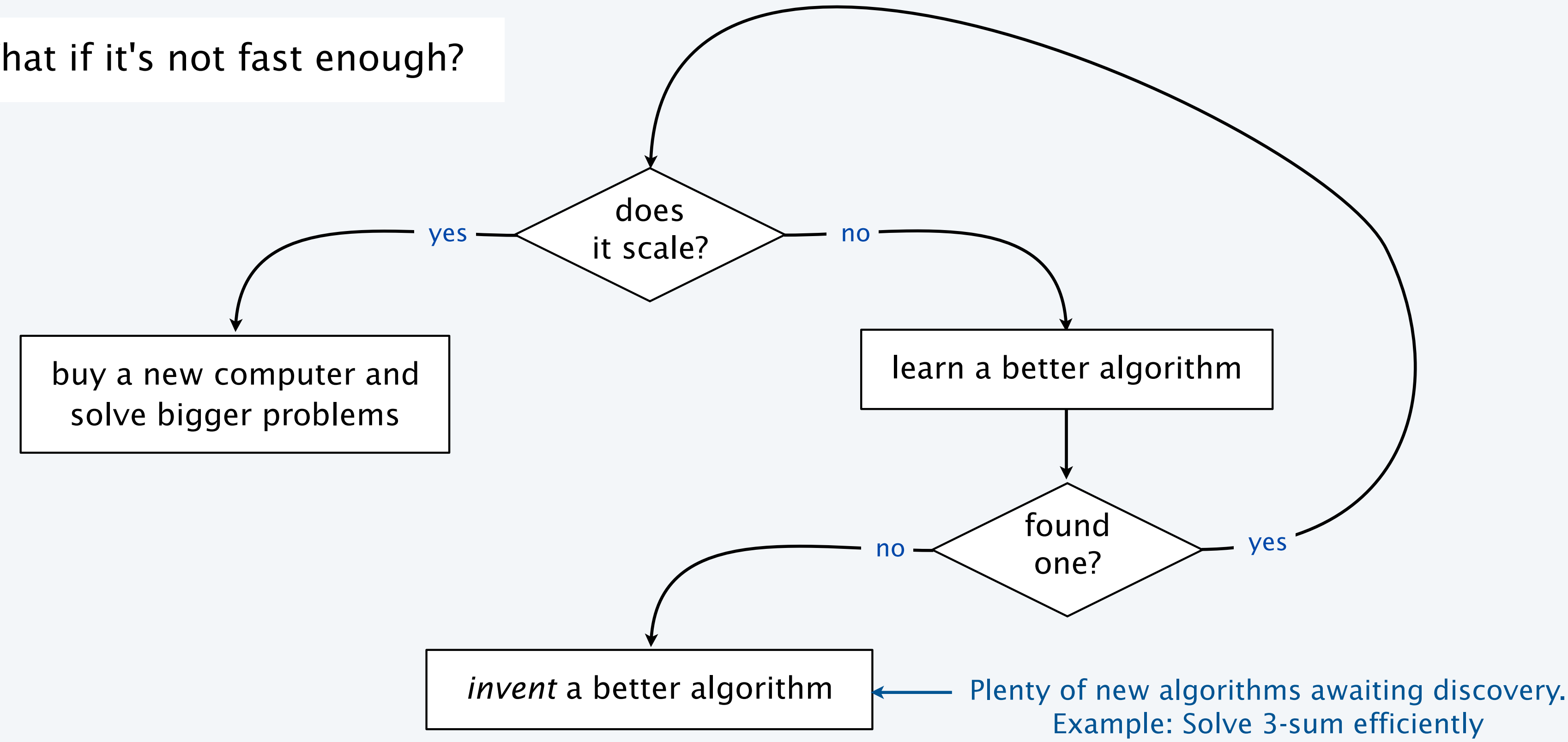
Summary

Use computational experiments, mathematical analysis, and the *scientific method* to learn whether your program might be useful to solve a large problem.

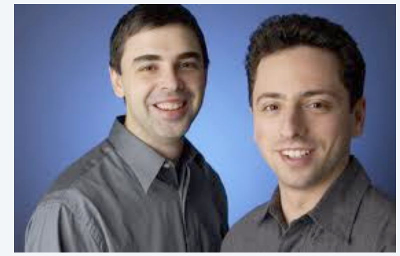
Q. What if it's not fast enough?

A.

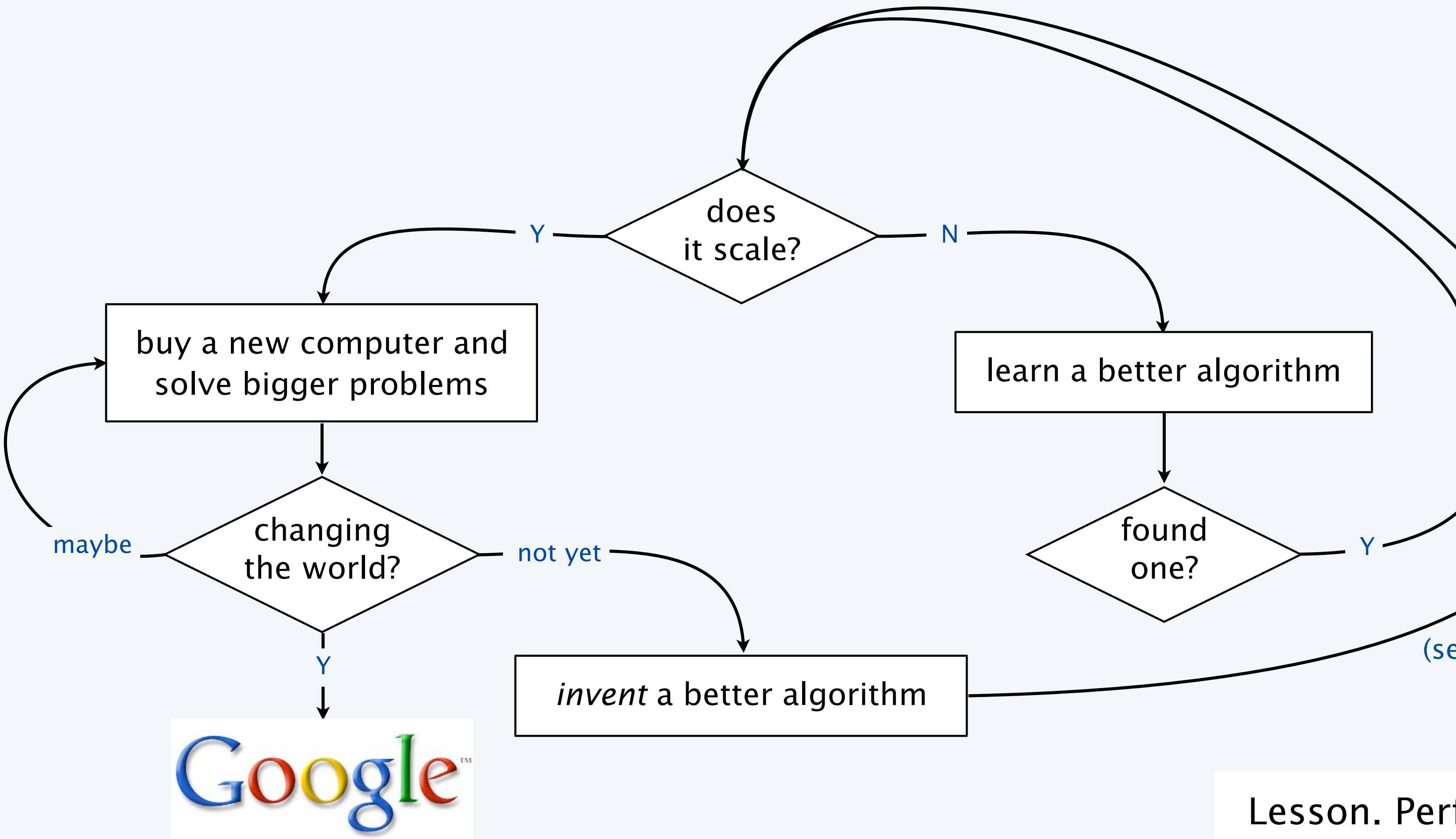


Case in point

Not so long ago, two CS grad students had a program to index the web (so as to enable search).



Larry Page and Sergey Brin



Lesson. Performance matters!

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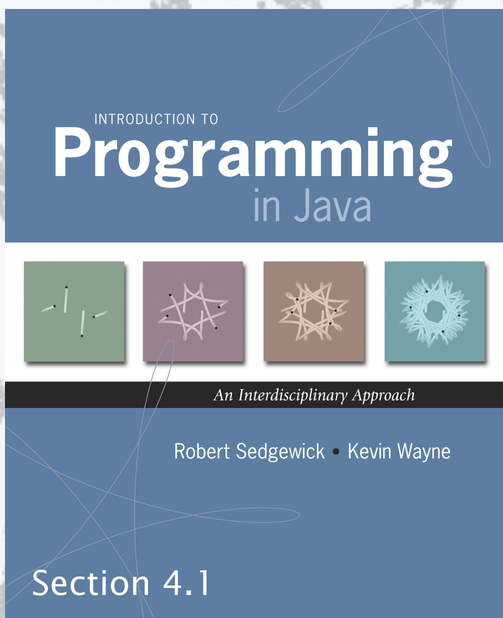

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