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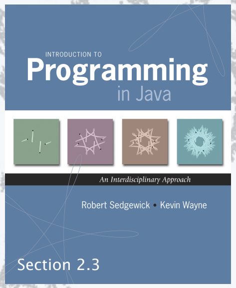

INTRODUCTION TO
Programming
in Java
An Interdisciplinary Approach
Robert Sedgewick · Kevin Wayne

Section 2.3

http://introcs.cs.princeton.edu

7. Recursion

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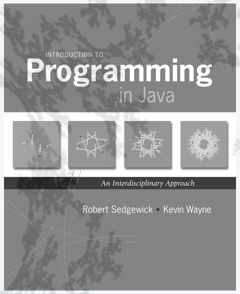

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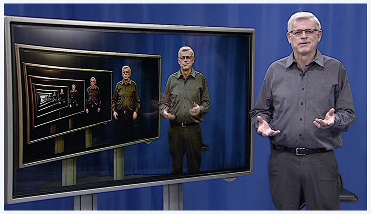
7. Recursion

- **Fundamentals**
- A classic example
- Recursive graphics
- Avoiding exponential waste
- Dynamic programming

Overview



Q. What is recursion?

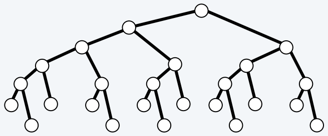
A. When something is specified in terms of *itself*.

Why learn recursion?

- Represents a new mode of thinking.
- Provides a powerful programming paradigm.
- Enables reasoning about correctness.
- Gives insight into the nature of computation.

Many computational artifacts are *naturally* self-referential.

- File system with folders containing folders.
- Fractal graphical patterns.
- Divide-and-conquer algorithms (stay tuned).



Example: Convert an integer to binary

Recursive program

To compute a function of N

- Base case. Return a value small N .
- Reduction step. Assuming that it works for smaller values of its argument, use the function to compute a return value for N.

```
public class Binary
{
   public static String convert(int N)
   {
      if (N == 1) return "1";
      return convert(N/2) + (N % 2);
   }
   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      System.out.println(convert(N));
   }
}
```

```
% java Binary 6
110
% java Binary 37
100101
% java Binary  999999
11110100001000111111
```

Q. How can we be convinced that this method is correct?

A. Use *mathematical induction.*

Mathematical induction (quick review)

To prove a statement involving N

- Base case. Prove it for some specific values of N.
- Induction step. Assuming that the statement is true for all positive integers less than N, use that fact to prove it for N.

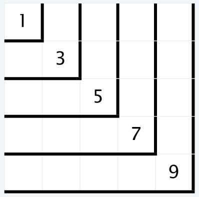
Example

The sum of the first N odd integers is N^2.

Base case. True for $N = 1$.

Induction step. The Nth odd integer is $2N - 1$. Let $T_N = 1 + 3 + 5 + ... + (2N - 1)$ be the sum of the first N odd integers.

- Assume that $T_{N-1} = (N - 1)^2$.
- Then $T_N = (N - 1)^2 + (2N - 1) = N^2$.



An alternate proof

Proving a recursive program correct

Recursion

To compute a function of N

- Base case. Return a value for small N .
- Reduction step. Assuming that it works for smaller values of its argument, use the function to compute a return value for N.

Mathematical induction

To prove a statement involving N

- Base case. Prove it for small N.
- Induction step. Assuming that the statement is true for all positive integers less than N, use that fact to prove it for N.

Recursive program

```
public static String convert(int N)
{
   if (N == 1) return "1";
   return convert(N/2) + (N % 2);
}
```

Correctness proof, by induction

`convert()` computes the binary representation of N

- Base case. Returns "1" for $N = 1$.
- Induction step. Assume that `convert()` works for $N/2$
  1. Correct to append "0" if N is even, since $N = 2(N/2)$.

  2. Correct to append "1" if N is odd since $N = 2(N/2) + 1$.

Mechanics of a function call

System actions when *any* function is called

- *Save environment* (values of all variables).
- *Initialize values* of argument variables.
- *Transfer control* to the function.
- *Restore environment* (and assign return value)
- *Transfer control* back to the calling code.

```
public class Binary
{
   public static String convert(int N)
   {
      if (N == 1) return "1";
      return convert(N/2) + (N % 2);
   }
   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      System.out.println(convert(N));
   }
}
```

```
% java Convert 26
11010
```

```
convert(26)
   if (N == 1) return "1";
   return   "1101"    + "0";
```

```
convert(13)
   if (N == 1) return "1";
   return    "110"    + "1";
```

```
convert(6)
   if (N == 1) return "1";
   return     "11"    + "0";
```

```
convert(3)
   if (N == 1) return "1";
   return      "1"    + "1";
```

```
convert(1)
   if (N == 1) return "1";
   return convert(0) + "1";
```

Programming with recursion: typical bugs

Missing base case

```
public static double bad(int N)
{
   return bad(N-1) + 1.0/N;
}
```

No convergence guarantee

```
public static double bad(int N)
{
   if (N == 1) return 1.0;
   return bad(1 + N/2) + 1.0/N;
}
```

Try *N* = 2

Both lead to *infinite recursive loops* (bad news).

need to know how to stop them on your computer

Collatz Sequence

Collatz function of *N*.
- If *N* is 1, stop.
- If *N* is even, divide by 2.
- If *N* is odd, multiply by 3 and add 1.

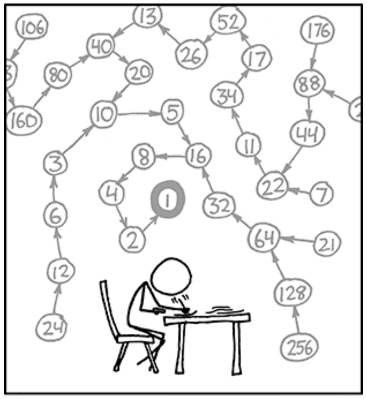
7	22	11	34	17	52	26	13	49	20	...

```
public static void collatz(int N)
{
    Stdout.print(N + " ");
    if (N == 1) return;
    if (N % 2 == 0) collatz(N / 2);
    else collatz(3*N + 1);
}
```

```
% java Collatz 7
7 22 11 34 17 52 26 13 40 20 10 5 16 8 4 2 1
```

Amazing fact. No one knows whether or not this function terminates for all *N* (!)

Note. We usually ensure termination by only making recursive calls for smaller *N*.



THE COLLATZ CONJECTURE STATES THAT IF YOU PICK A NUMBER, AND IF IT'S EVEN DIVIDE IT BY TWO AND IF IT'S ODD MULTIPLY IT BY THREE AND ADD ONE, AND YOU REPEAT THIS PROCEDURE LONG ENOUGH, EVENTUALLY YOUR FRIENDS WILL STOP CALLING TO SEE IF YOU WANT TO HANG OUT.

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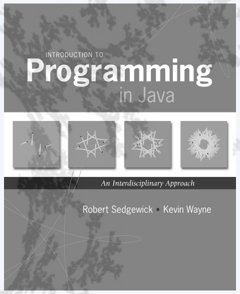

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7. Recursion

- **Fundamentals**
- A classic example
- Recursive graphics
- Avoiding exponential waste
- Dynamic programming

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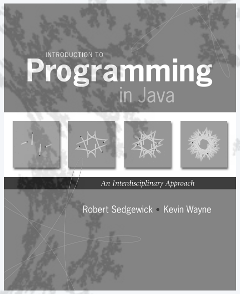

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7. Recursion

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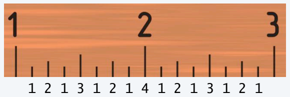
Warmup: subdivisions of a ruler (revisited)

`ruler(n)`: create subdivisions of a ruler to $1/2^n$ inches.

- Return one space for $n = 0$.
- Otherwise, sandwich n between two copies of `ruler(n-1)`.



```
public class RulerR
{
   public static String ruler(int n)
   {
      if (n == 0) return " ";
      return ruler(n-1) + n + ruler(n-1);
   }
   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      StdOut.println(ruler(n));
   }
}
```

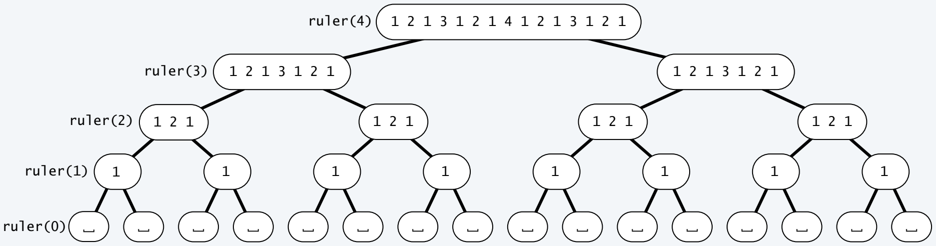
```
% java RulerR 1
 1
% java RulerR 2
 1 2 1
% java RulerR 3
 1 2 1 3 1 2 1
% java RulerR 4
 1 2 1 3 1 2 1 4 1 2 1 3 1 2 1
% java RulerR 50
Exception in thread "main"
java.lang.OutOfMemoryError:
Java heap space
```

↑ $2^{50} - 1$ strings in output.

Tracing a recursive program

Use a *recursive call tree*

- One node for each recursive call.
- Label node with return value after children are labelled.



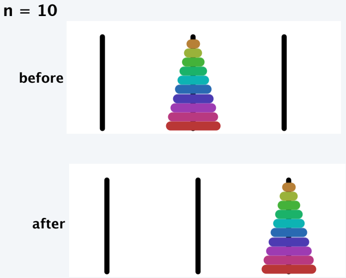
Towers of Hanoi puzzle

A legend of uncertain origin

- $n = 64$ discs of differing size; 3 posts; discs on one of the posts from largest to smallest.
- An ancient prophecy has commanded monks to move the discs to another post.
- When the task is completed, *the world will end.*

Rules

- Move discs one at a time.
- Never put a larger disc on a smaller disc.



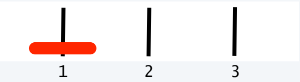
Q. Generate list of instruction for monks ?

Q. When might the world end ?

Towers of Hanoi

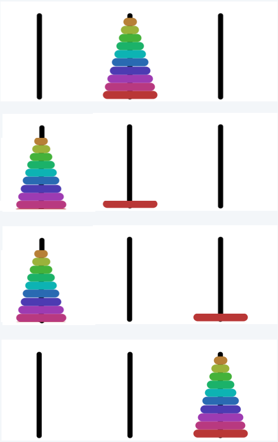
For simple instructions, use cyclic wraparound

- Move *right* means 1 to 2, 2 to 3, or 3 to 1.
- Move *left* means 1 to 3, 3 to 2, or 2 to 1.

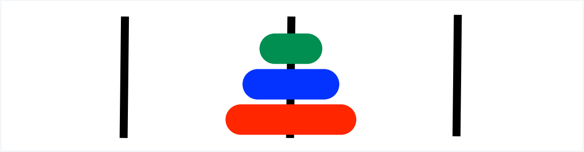


A recursive solution

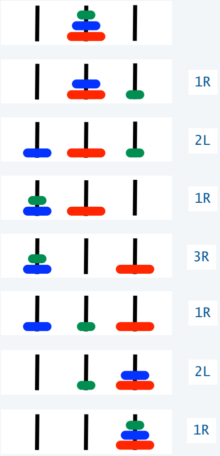
- Move *n* – 1 discs to the left (recursively).
- Move largest disc to the *right*.
- Move *n* – 1 discs to the left (recursively).



Towers of Hanoi solution (n = 3)



1R	2L	1R	3R	1R	2L	1R



Towers of Hanoi: recursive solution

`hanoi(n)`: Print moves for *n* discs.

- Return one space for *n* = 0.
- Otherwise, set `move` to the specified move for disc *n*.
- Then sandwich `move` between two copies of `hanoi(n-1)`.

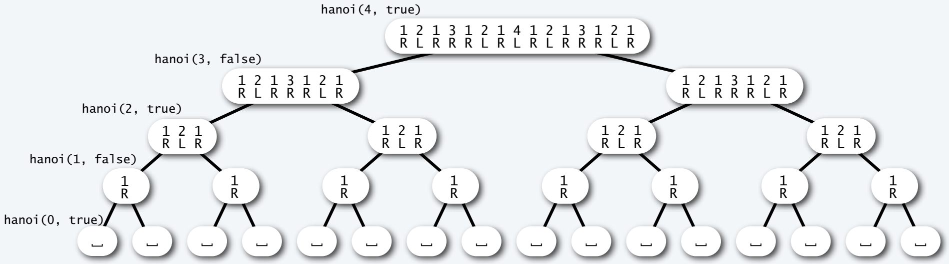
```
public class HanoiR
{
   public static String hanoi(int n, boolean left)
   {
      if (n == 0) return " ";
      String move;
      if (left) move = n + "L";
      else      move = n + "R";
      return hanoi(n-1, !left) + move + hanoi(n-1, !left);
   }
   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      StdOut.println(hanoi(n, false));
   }
}
```

```
% java HanoiR 3
 1R 2L 1R 3R 1R 2L 1R
```

Recursive call tree for towers of Hanoi

Structure is the *same* as for the ruler function and suggests 3 useful and easy-to-prove facts.

- Each disc always moves in the same direction.
- Moving smaller disc always alternates with a unique legal move.
- Moving *n* discs requires $2^n - 1$ moves.



Answers for towers of Hanoi

Q. Generate list of instructions for monks ?

A. (Long form). 1L 2R 1L 3L 1L 2R 1L 4R 1L 2R 1L 3L 1L 2R 1L 5L 1L 2R 1L 3L 1L 2R 1L 4R ...

A. (Short form). Alternate "1L" with the only legal move not involving the disc 1.

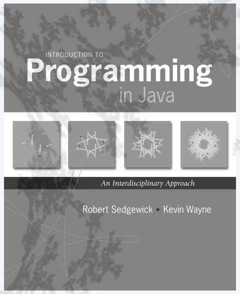
"L" or "R" depends on whether *N* is odd or even

Q. When might the world end ?

A. Not soon: need $2^{64} - 1$ moves.

Note: Recursive solution has been proven optimal.

moves per second	*end of world*
1	5.84 billion centuries
1 billion	5.84 centuries



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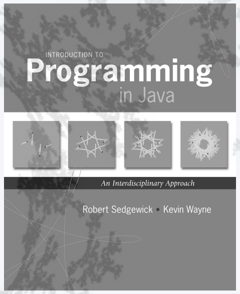

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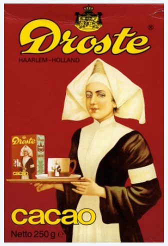
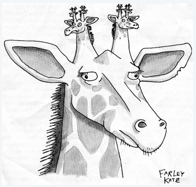
Recursive graphics in the wild





WEEKEND Arts

The New York Times

Fruits of Design, Certified Organic

The Gifts to Open Again and Again

Black, White and Read All Over Over

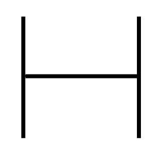
Divine and Devotee Meet Across Hinges

"Hello, World" of recursive graphics: H-trees

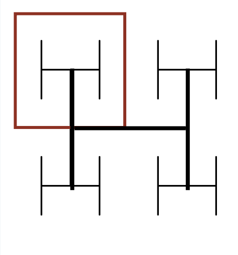
H-tree of order *n*

- If *n* is 0, do nothing.
- Draw an H, centered.
- Draw four H-trees of order *n* −1 and half the size, centered at the tips of the H.

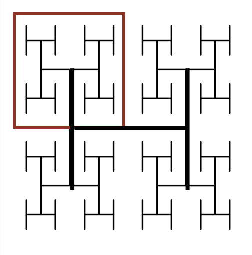
order 1



order 2

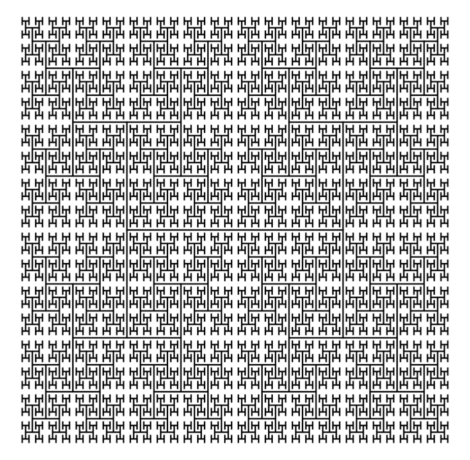


order 3



H-trees

Application. Connect a large set of regularly spaced sites to a single source.



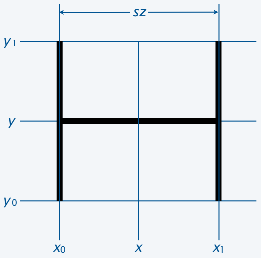
order 6

Recursive H-tree implementation

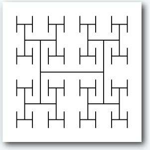
```
public class Htree
{
   public static void draw(int n, double sz, double x, double y)
   {
      if (n == 0) return;
      double x0 = x - sz/2, x1 = x + sz/2;
      double y0 = y - sz/2, y1 = y + sz/2;
      StdDraw.line(x0,  y, x1,  y);
      StdDraw.line(x0, y0, x0, y1);
      StdDraw.line(x1, y0, x1, y1);
      draw(n-1, sz/2, x0, y0);
      draw(n-1, sz/2, x0, y1);
      draw(n-1, sz/2, x1, y0);
      draw(n-1, sz/2, x1, y1);
   }
   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      draw(n, .5, .5, .5);
   }
}
```

draw the H, centered on (*x*, *y*)

draw four half-size H-trees



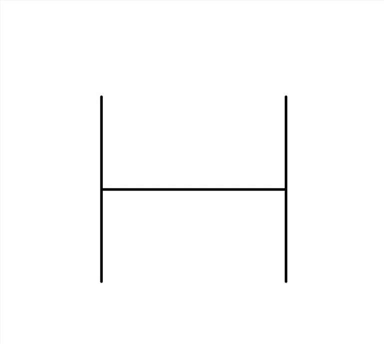
```
% java Htree 3
```



Deluxe H-tree implementation

```
public class HtreeDeluxe
{
   public static void draw(int n, double sz,
                                  double x, double y)
   {
      if (n == 0) return;
      double x0 = x - sz/2, x1 = x + sz/2;
      double y0 = y - sz/2, y1 = y + sz/2;
      StdDraw.line(x0,  y, x1,  y);
      StdDraw.line(x0, y0, x0, y1);
      StdDraw.line(x1, y0, x1, y1);
      StdAudio.play(PlayThatNote.note(n, .25*n));
      draw(n-1, sz/2, x0, y0);
      draw(n-1, sz/2, x0, y1);
      draw(n-1, sz/2, x1, y0);
      draw(n-1, sz/2, x1, y1);
   }
   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      draw(n, .5, .5, .5);
   }
}
```

```
% java HtreeDeluxe 4
```



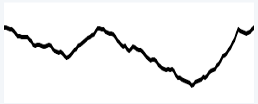
Fractional Brownian motion

A process that models many phenomenon.

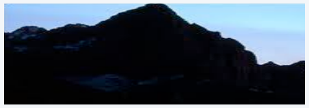
- Price of stocks.
- Dispersion of fluids.
- Rugged shapes of mountains and clouds.
- Shape of nerve membranes.

. . .

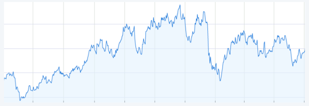
Brownian bridge model



An actual mountain



Price of an actual stock

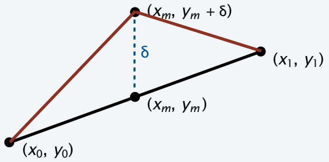


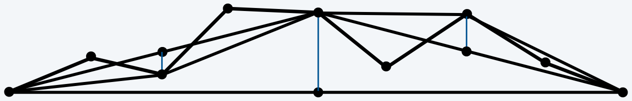
Black-Scholes model (two different parameters)

Fractional Brownian motion simulation

Midpoint displacement method

- Consider a line segment from (x_0, y_0) to (x_1, y_1).
- If sufficiently short draw it *and return*
- Divide the line segment in half, at (x_m, y_m).
- Choose δ at random *from Gaussian distribution.*
- Add δ to y_m.
- Recur on the left and right line segments.

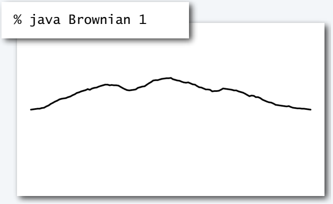




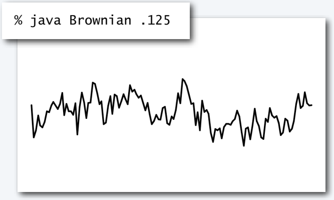
Brownian motion implementation

```
public class Brownian
{
   public static void
   curve(double x0, double y0, double x1, double y1,
                              double var, double s)
   {
      if (x1 - x0 < .01)
      {  StdDraw.line(x0, y0, x1, y1); return;  }
      double xm = (x0 + x1) / 2;
      double ym = (y0 + y1) / 2;
      double stddev = Math.sqrt(var);
      double delta = StdRandom.gaussian(0, stddev);
      curve(x0, y0, xm, ym+delta, var/s, s);
      curve(xm, ym+delta, x1, y1, var/s, s);
   }
   public static void main(String[] args)
   {
      double H = Double.parseDouble(args[0]);
      double s = Math.pow(2, 2*H);
      curve(0, .5, 1.0, .5, .01, s);
   }
}
```

control parameter
(see text)



```
% java Brownian 1
```

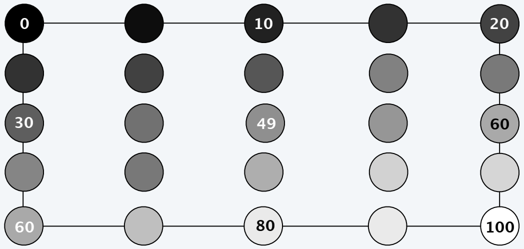


```
% java Brownian .125
```

A 2D Brownian model: plasma clouds

Midpoint displacement method

- Consider a rectangle centered at (x, y) with pixels at the four corners.
- If the rectangle is small, do nothing.
- Color the midpoints of each side the average of the endpoint colors.
- Choose δ at random *from Gaussian distribution.*
- Color the center pixel the average of the four corner colors *plus* δ
- Recurse on the four quadrants.



Reference: http://www.geocities.com/aaron_torpy/gallery.htm

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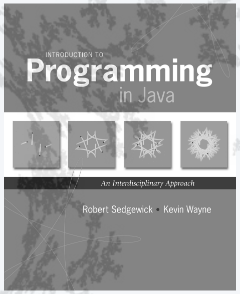

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7. Recursion

- Fundamentals
- A classic example
- **Recursive graphics**
- Avoiding exponential waste
- Dynamic programming

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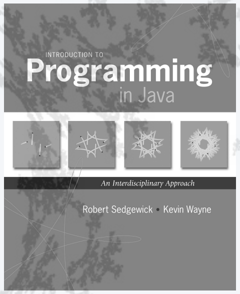

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Fibonacci numbers

Let $F_n = F_{n-1} + F_{n-1}$ for $n > 1$ with $F_0 = 0$ and $F_1 = 1$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	…
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	…

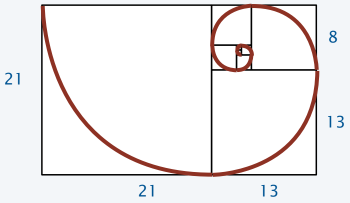
Models many natural phenomena and is widely found in art and architecture.

Leonardo Fibonacci
c. 1170 – c. 1250

Examples.

- Model for reproducing rabbits.
- Nautilus shell.
- Mona Lisa.
- …

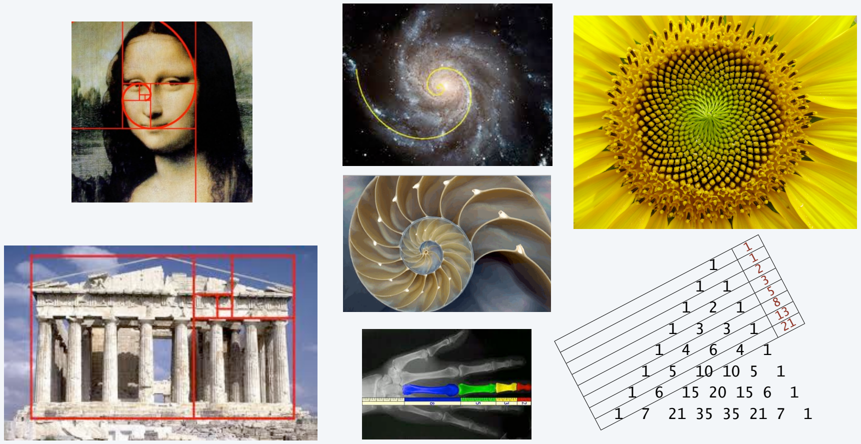
golden ratio F_n / F_{n-1}



Facts (known for centuries).

- F_n / F_{n-1} ➛ Φ = 1.618... as n ➛ ∞
- F_n is the closest integer to $\Phi^n/\sqrt{5}$

Fibonacci numbers and the golden ratio in the wild



Computing Fibonacci numbers

Q. [Curious individual.] What is the exact value of F_{50} ?

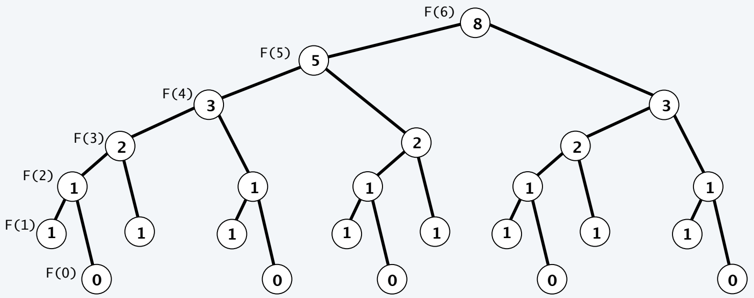
A. [Novice programmer.] Just a second. I'll write a recursive program to compute it.

```
public class FibonacciR
{
   public static long F(int n)
   {
      if (n == 0) return 0;
      if (n == 1) return 1;
      return F(n-1) + F(n-2);
   }
   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      StdOut.println(F(n));
   }
}
```

```
% java FibonacciR 5
5
% java FibonacciR 6
8
% java FibonacciR 10
55
% java FibonacciR 12
144
% java FibonacciR 50
12586269025
```

takes a few minutes
Hmmm. Why is that?

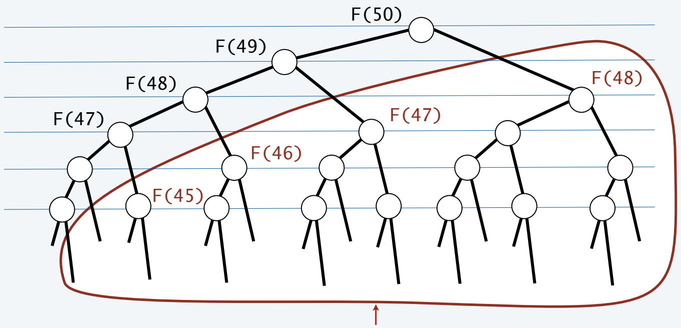
Recursive call tree for Fibonacci numbers



Exponential waste

Let C_n be the number of times F(n) is called when computing F(50).

n	C_n	
50	1	F_1
49	1	F_2
48	2	F_3
47	3	F_4
46	5	F_5
45	8	F_6
	…	…
0	$>10^{10}$	F_{51}



Exponentially wasteful to recompute all these values.
(10^{10} calls on F(0), not to mention calls on F(1), F(2),...)

Exponential waste dwarfs progress in technology.

If you engage in exponential waste, you *will not* be able to solve a large problem.

1970s


VAX 11/780

n	*time to compute* F_n
30	minutes
40	hours
50	weeks
60	years
70	centuries
80	millenia

2010s: 10,000+ times faster


Macbook Air

n	*time to compute* F_n
50	minutes
60	hours
70	weeks
80	years
90	centuries
100	millenia

1970s: "That program won't compute F_{60} before you graduate! "

2010s: "That program won't compute F_{80} before you graduate! "

Avoiding exponential waste

Memoization

- Maintain an array `memo[]` to remember all computed values.
- If value known, just return it.
- Otherwise, compute it, remember it, and then return it.

```
public class FibonacciM
{
   static long[] memo =  new long[100];
   public static long F(int N)
   {
      if (n == 0) return 0;
      if (n == 1) memo[1] = 1;
      if (memo[n] != 0) return memo[n];
      memo[n] = F(n-1) + F(n-2);
      return memo[n];
   }
   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      StdOut.println(F(n));
   }
}
```

```
% java FibonacciM 50
12586269025
% java FibonacciM 60
1548008755920
% java FibonacciM 80
23416728348467685
```

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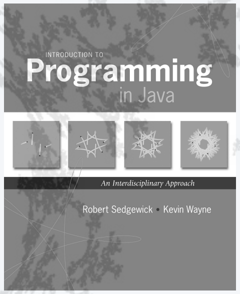

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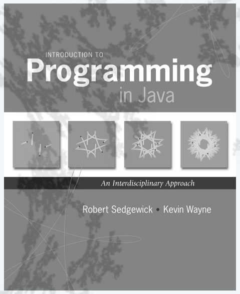

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An efficient alternative to recursion

Dynamic programming.
- Build computation from the "*bottom up*".
- Solve small subproblems *and save solutions*.
- Use those solutions to build bigger solutions.


Richard Bellman
1920-1984

Fibonacci numbers

```
public class Fibonacci
{
   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      long[] F = new long[n+1];
      F[0] = 0; F[1] = 1;
      for (int i = 2; i <= n; i++)
         F[i] =  F[i-1] + F[i-2];
      StdOut.println(F[n]);
   }
}
```

```
% java FibonacciM 50
12586269025
% java FibonacciM 60
1548008755920
% java FibonacciM 80
23416728348467685
```

Key advantage over recursive solution. Each subproblem is addressed only *once*.

How many ways to change a dollar?

Q. How many ways to change a dollar with quarters ?

A. **1**

Q. How many ways to change a dollar with quarters *and dimes*?

A. **3**

Q. How many ways to change a dollar with quarters, dimes *and nickels*?

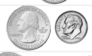
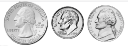
Q. How many ways to change a dollar with quarters, dimes, nickels *and pennies*?

How many ways to change a dollar?

Dynamic programming solution (Pólya).
- Count 1 way to change 0 cents.
- Maintain an array change[] for the number of known ways so far.
- For each coin V, pass through and update the array:

```
for (int k = V; k <= N; k++) a[k] += a[k-V];
```


George Polya
1887-1985

	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
	1					1					1					1					1
	1		1		1	1	1	1	1	1	2	1	2	1	2	2	2	2	2	2	3
	1	1	2	2	3	4	5	6	7	8	10	11	13	14	16	18	20	22	24	26	29
	1	2	4	6	9	13	18	24	31	39	49	60	73	87	103	121	141	163	187	213	242

How many ways to change a dollar?

Dynamic programming solution

```
public class Change
{
   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      int[] a = new int[N+1];
      a[0] = 1;
      for (int k = 25; k<=N; k++) a[k] += a[k-25];
      for (int k = 10; k<=N; k++) a[k] += a[k-10];
      for (int k =  5; k<=N; k++) a[k] += a[k- 5];
      for (int k =  1; k<=N; k++) a[k] += a[k- 1];
      StdOut.println(a[N]);
   }
}
```

```
% java Change 100
242
```

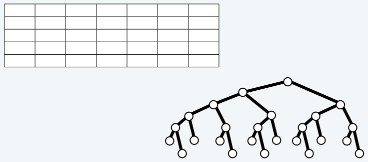
Note. Recursive solution is *much more complicated* and can be *exponentially wasteful.*

Dynamic programming and recursion

Broadly useful approaches to solving problems by combining solutions to smaller subproblems.

Why learn DP and recursion?

- Represent a new mode of thinking.
- Provide powerful programming paradigms.
- Give insight into the nature of computation.
- Successfully used for decades.



	recursion	*dynamic programming*
advantages	Decomposition often obvious. Easy to reason about correctness.	Avoids exponential waste. Often simpler than memoization.
pitfalls	Potential for exponential waste. Decomposition may not be simple.	Uses significant space. Not suited for real-valued arguments. Challenging to determine order of computation

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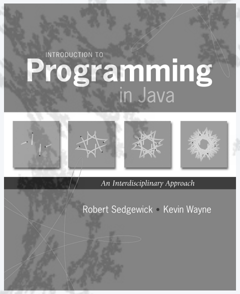

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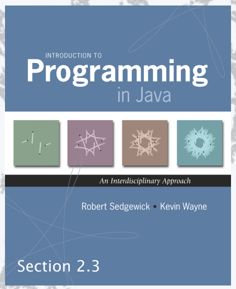

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