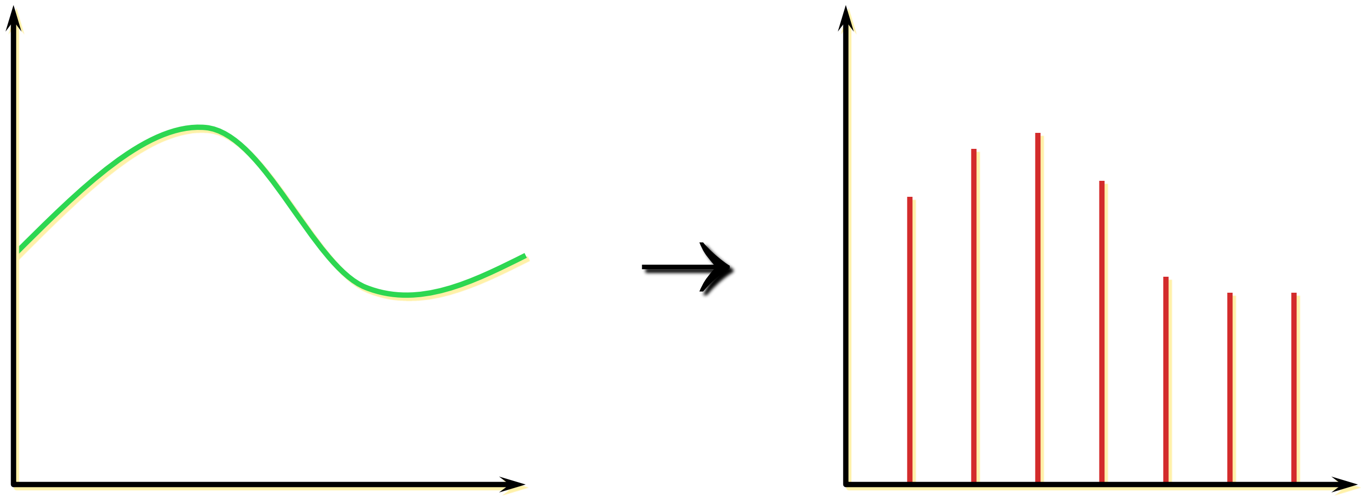
Sampling and Aliasing

COS 323

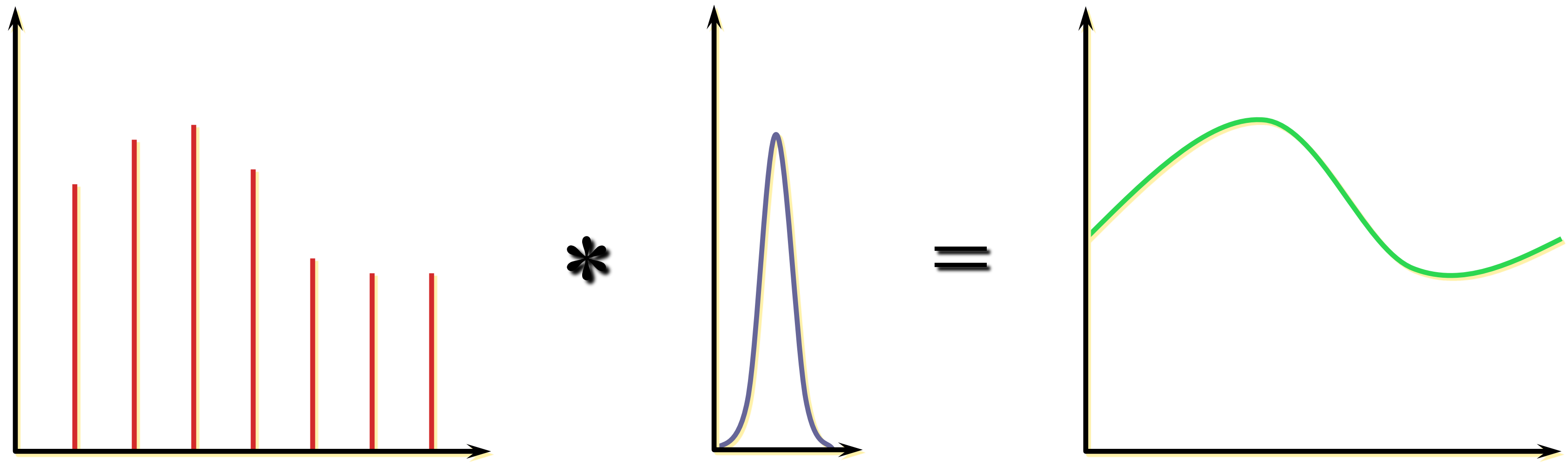
Signal Processing

- Sampling a continuous function



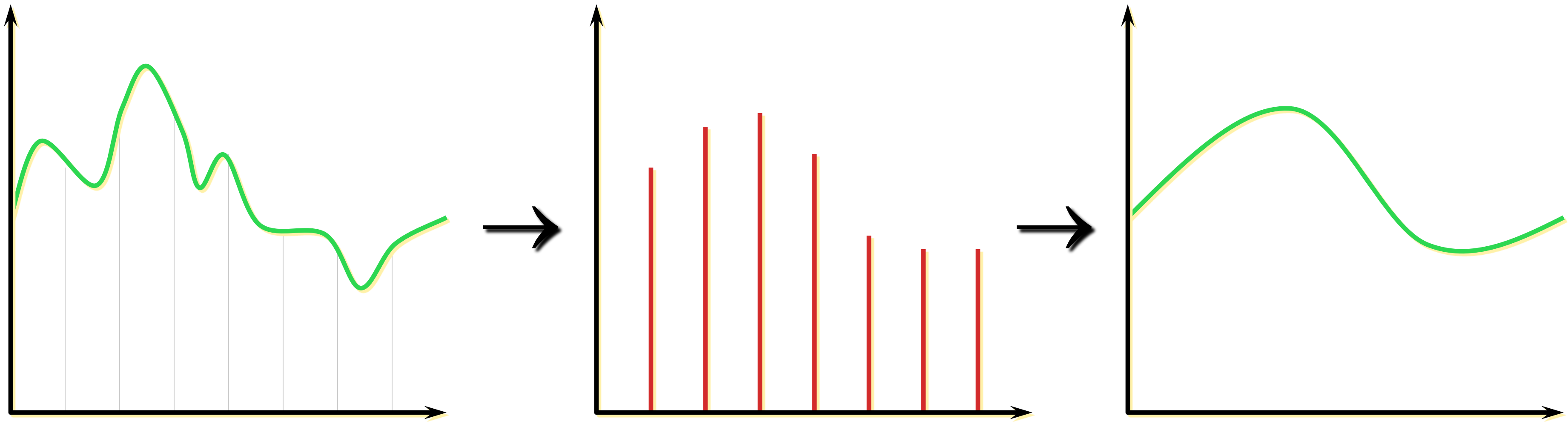
Signal Processing

- Convolve with reconstruction filter to re-create signal



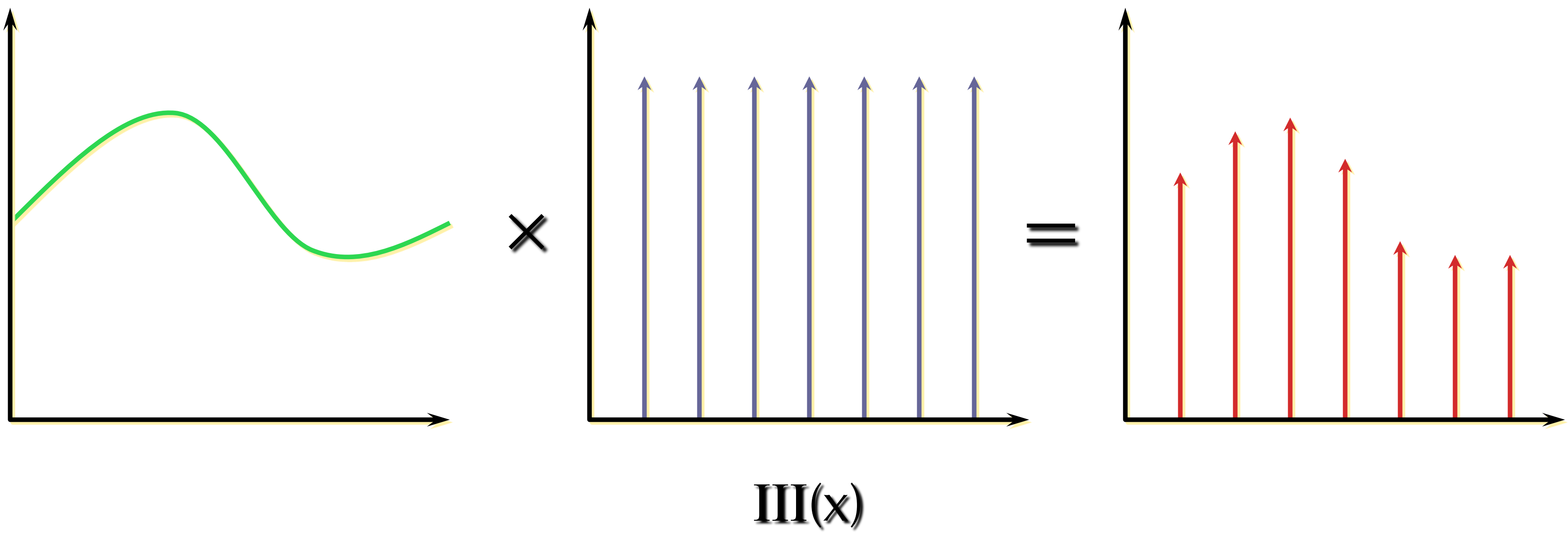
How to Sample?

- Reconstructed signal might be very different from original: "aliasing"



Why Does Aliasing Happen?

- Sampling = multiplication by shah function III(x) (also known as impulse train)



Digression: Delta Function

Kronecker delta

$$\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

Dirac delta

$$\delta(x) = \begin{cases} 0 \text{ if } x \neq 0 \\ \infty \text{ if } x = 0 \end{cases}$$

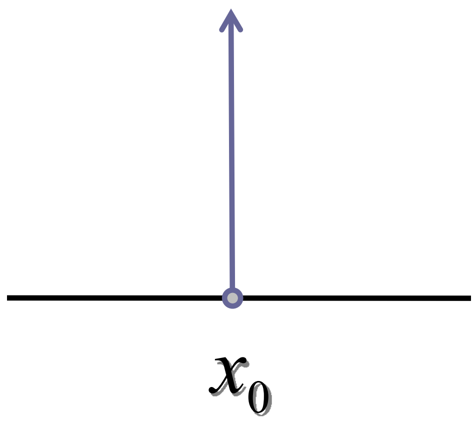
$$\int \delta(x)dx = 1$$

- Can think of as $$\delta(x) = \lim_{\sigma \to 0^+} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

Scaled and Translated Dirac Delta

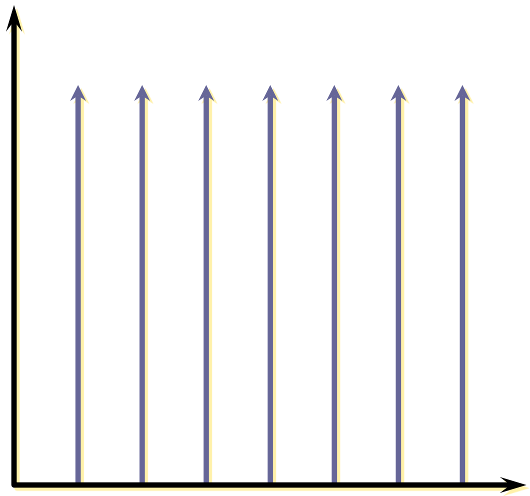
$$c\delta(x-x_0)=\begin{cases}0 \text{ if } x \neq x_0 \\ \infty \text{ if } x = x_0\end{cases}$$

$$\int c\delta(x-x_0)dx = c$$



Impulse Train

$$\mathrm{III}(x) = \cdots + \partial(x+2) + \partial(x+1) + \partial(x) + \partial(x-1) + \partial(x-2) + \cdots$$



III(x)

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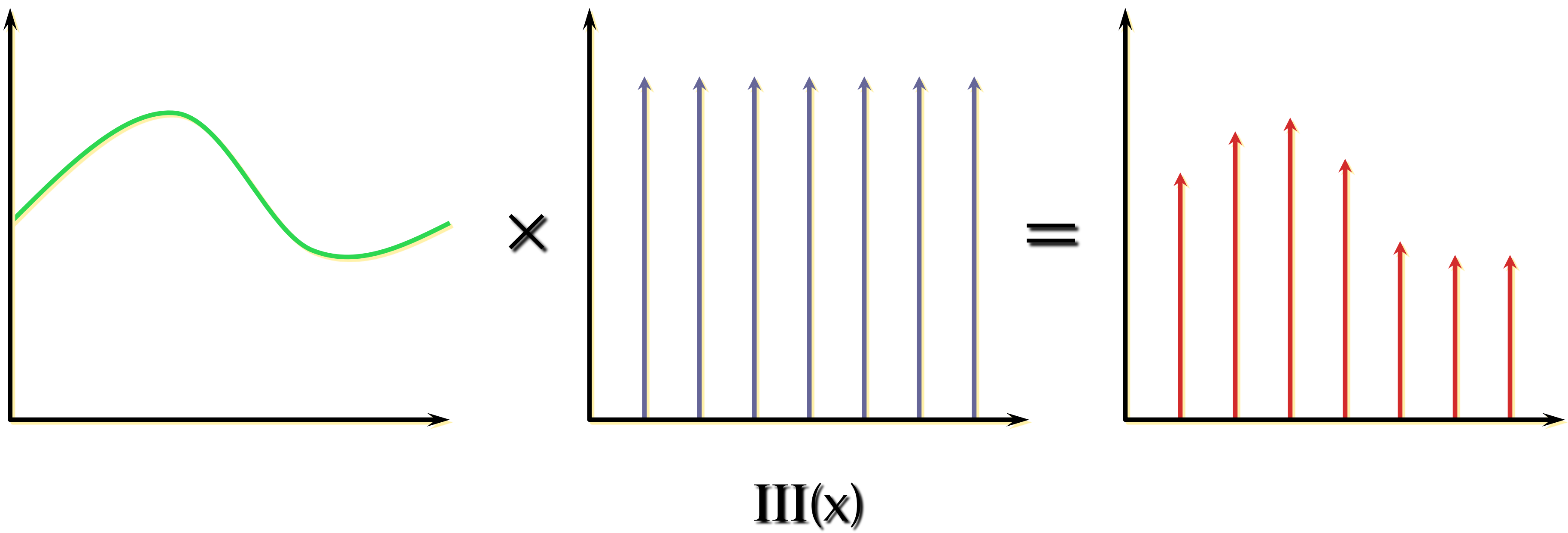
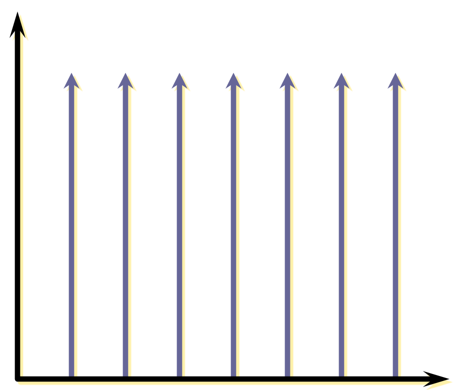


Fourier Analysis

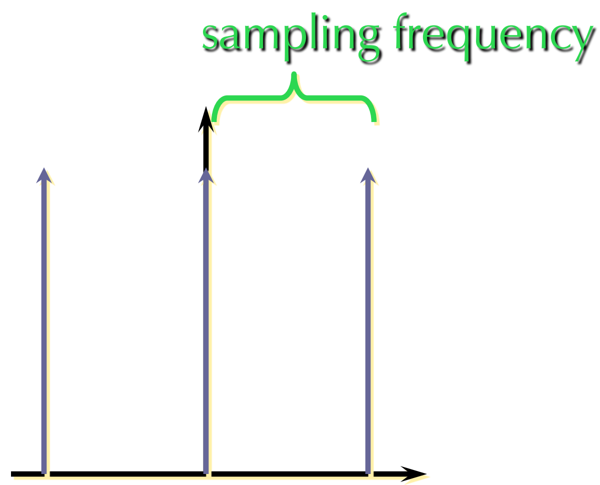
- Multiplication in primal space = convolution in frequency space

$$\mathcal{F}\big(f(x)g(x)\big)=\mathcal{F}\big(f(x)\big)*\mathcal{F}\big(g(x)\big)$$

- Fourier transform of III is III

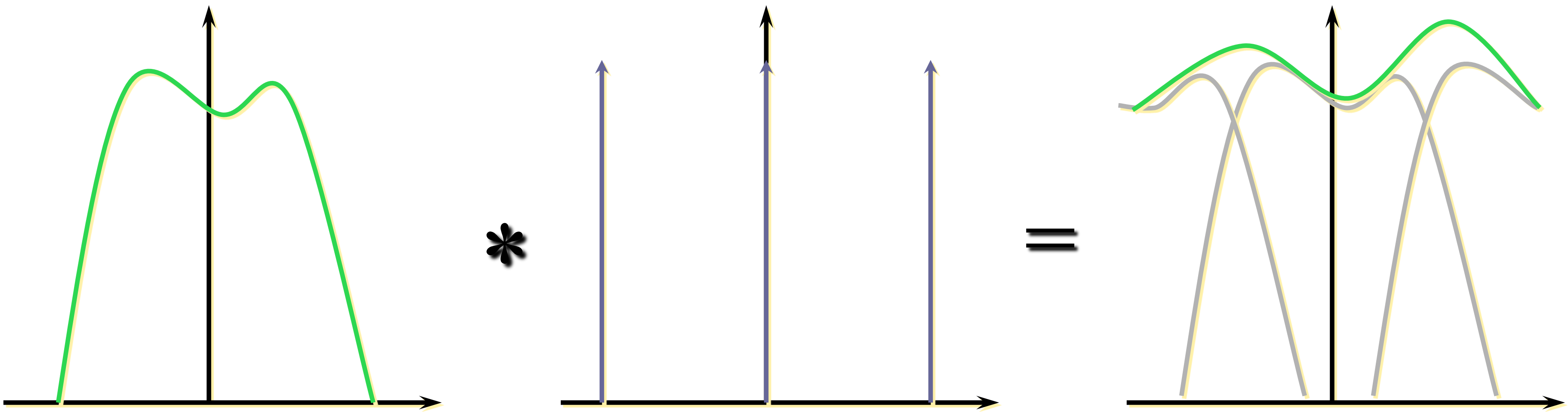


III(x)



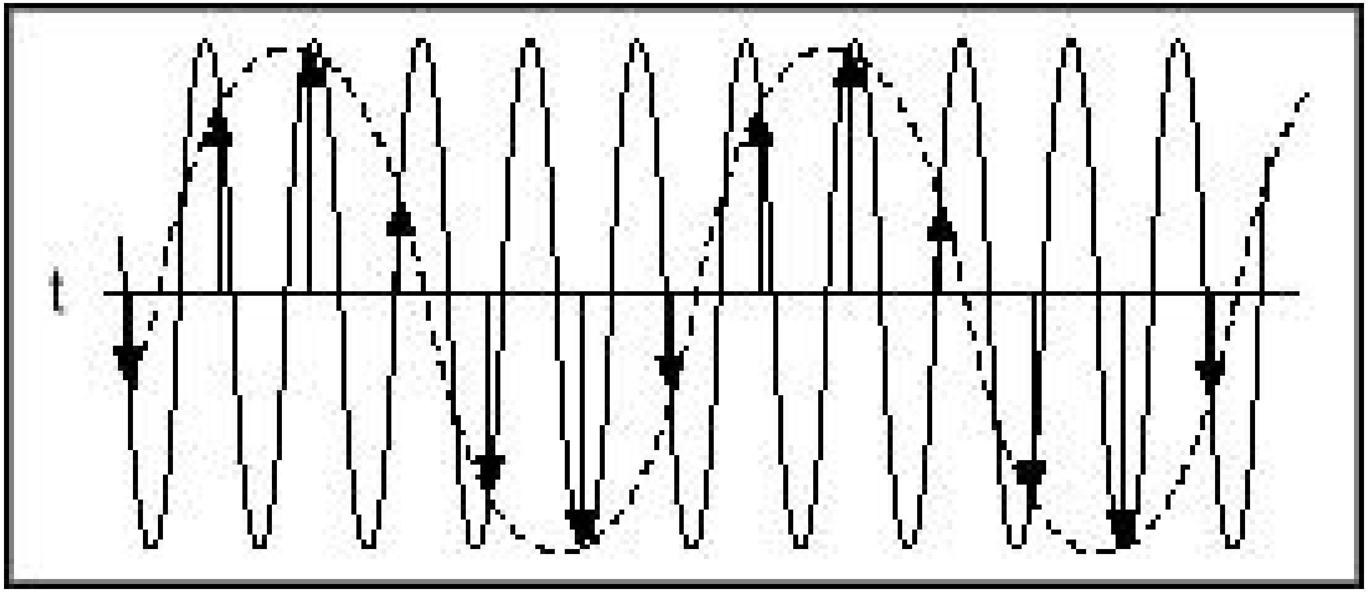
$\mathcal{F}(\text{III}(x))$

Fourier Analysis



- Result: high frequencies can "alias" into low frequencies

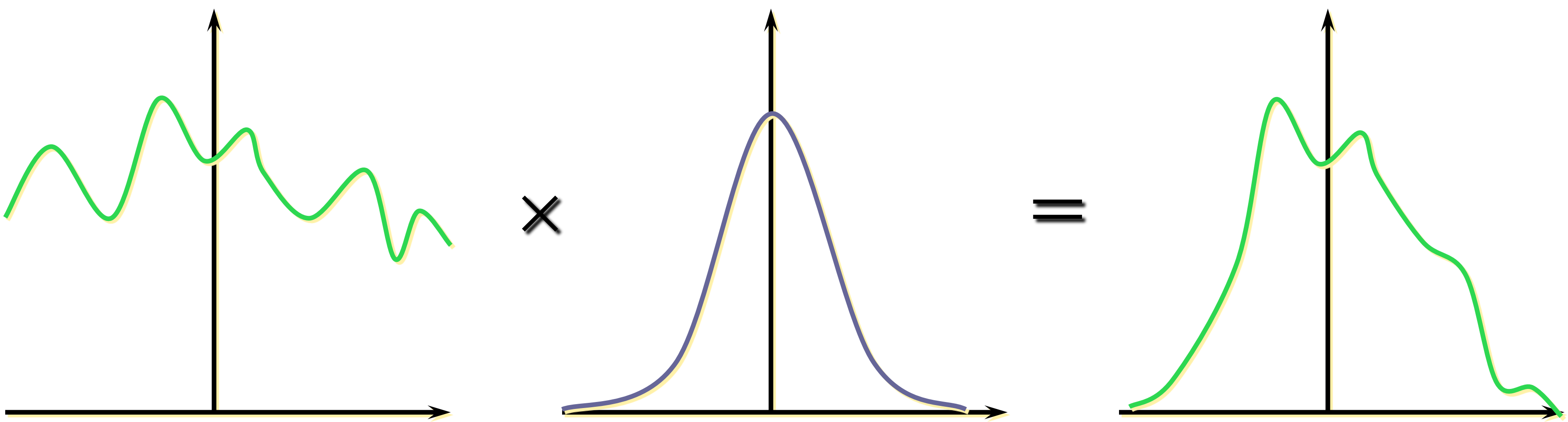
Aliasing



Fourier Analysis

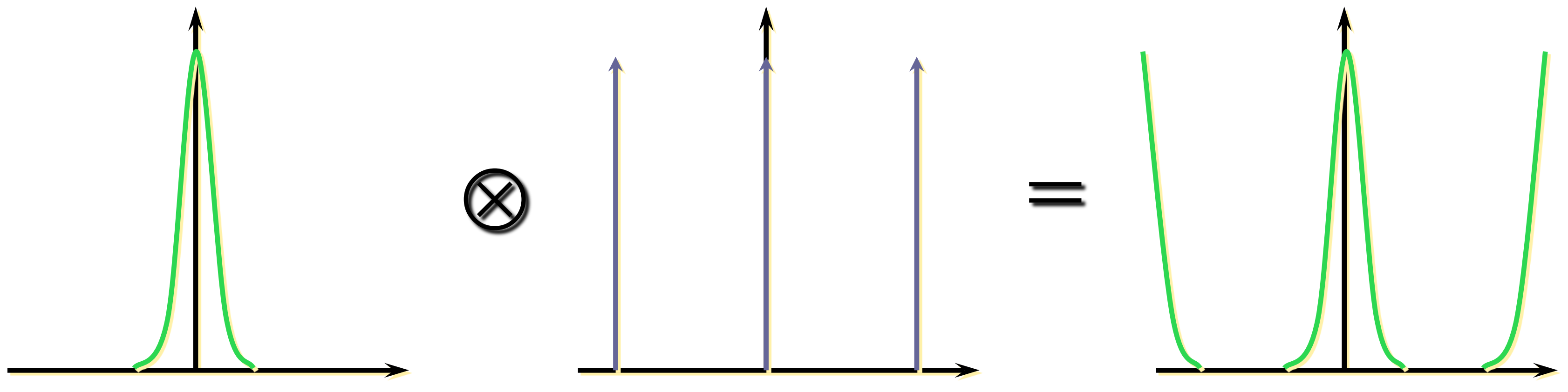
- Convolution with reconstruction filter = multiplication in frequency space

$$\mathcal{F}\big(f(x) * g(x)\big) = \mathcal{F}\big(f(x)\big)\mathcal{F}\big(g(x)\big)$$



Aliasing in Frequency Space

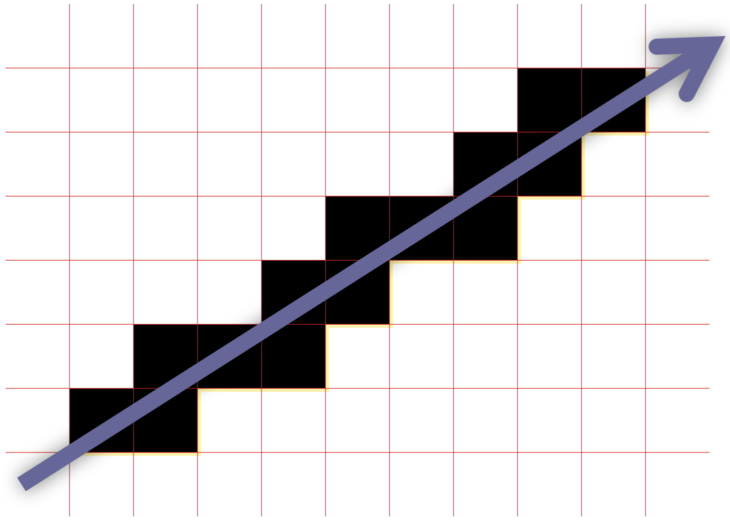
- Conclusions:
  - High frequencies can alias into low frequencies
  - Can't be cured by a different reconstruction filter
  - Nyquist limit: capture all frequencies iff bandlimited – maximum frequency < ½ sampling rate



Aliasing strikes!

Other Aliasing Examples

- Car wheel "spins backwards" on film
- Jaggies in graphics



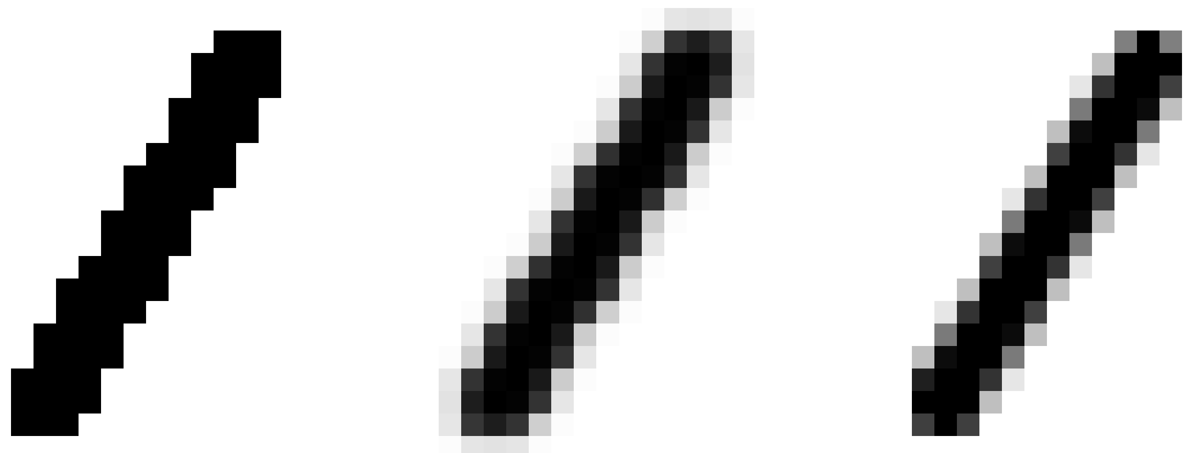
- "Crawling jaggies" on edges of objects as they move

Filters for Sampling

- Solution: insert filter *before* sampling
  - "Sampling" or "bandlimiting" or "antialiasing" filter

  - Low-pass filter
  - Eliminate frequency content above Nyquist limit
  - Result: aliasing replaced by blur
  - Partial alternative: oversampling, digital filtering

Antialiasing Jaggies



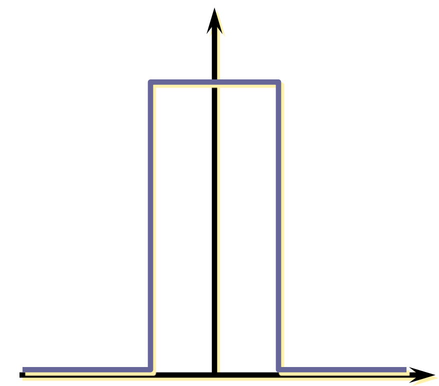
Aliased

Postfiltered: blurry jaggies

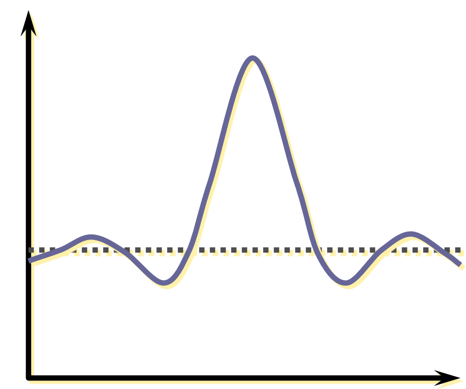
Correctly prefiltered

Ideal Sampling Filter

- “Brick wall” filter:
  box in frequency

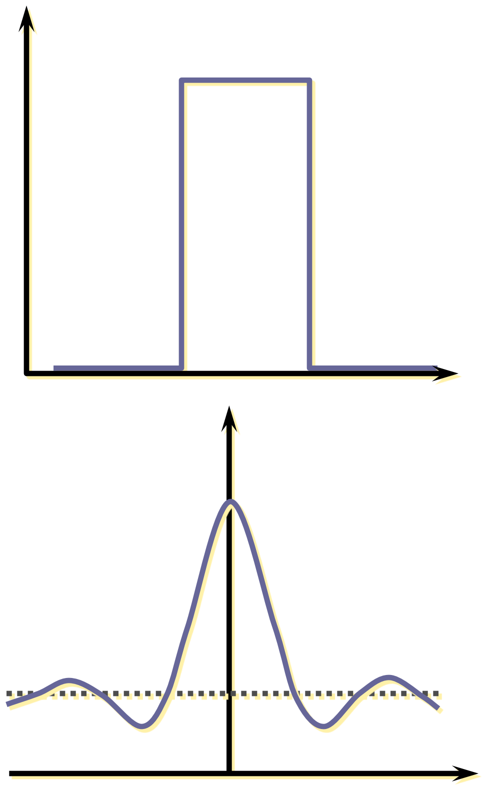


- In space: sinc function
  - sinc(x) = sin(x) / x
  - Infinite support
  - Possibility of “ringing”

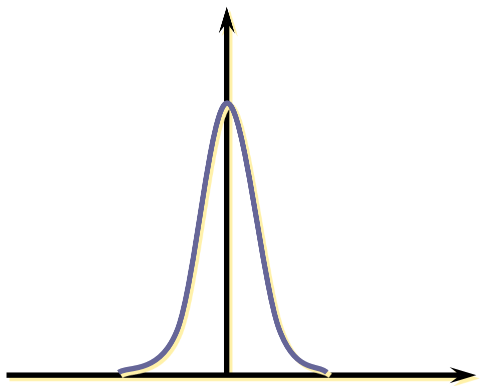


Cheap Sampling Filter

- Box in space
  - Cheap to evaluate
  - Finite support
- In frequency: sinc
  - Imperfect bandlimiting



Gaussian Sampling Filter



- Fourier transform of Gaussian = Gaussian
- Good compromise as sampling filter:
  - Well approximated by function w. finite support
  - Good bandlimiting performance