PDE Solver Stability and Multigrid Methods

Review of Finite Differences Method

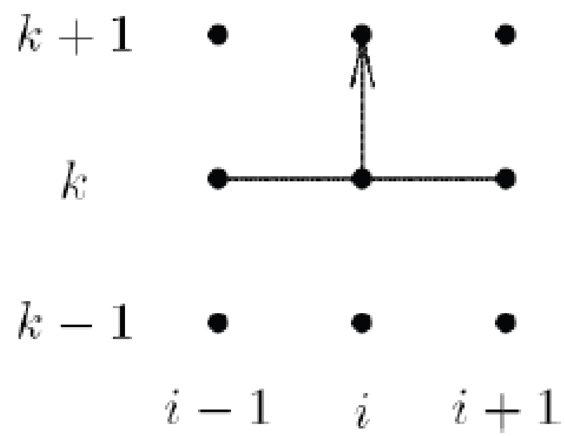
- Consider heat equation

$$u_t = c\, u_{xx}, \qquad 0 \le x \le 1, \qquad t \ge 0$$

with initial and boundary conditions

$$u(0, x) = f(x), \qquad u(t, 0) = \alpha, \qquad u(t, 1) = \beta$$

$$\frac{u_i^{k+1} - u_i^k}{\Delta t} = c\, \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{(\Delta x)^2}$$



Finite Differences vs. Method of Lines

- Finite difference method yields recurrence relation:

$$u_i^{k+1} = u_i^k + c\,\frac{\Delta t}{(\Delta x)^2}\left(u_{i+1}^k - 2u_i^k + u_{i-1}^k\right), \quad i = 1, \ldots, n$$

- Compare to method of lines with spatial mesh size Δx:

$$y_i'(t) = \frac{c}{(\Delta x)^2}\left(y_{i+1}(t) - 2y_i(t) + y_{i-1}(t)\right), \quad i = 1, \ldots, n$$

- Finite difference method is equivalent to solving each y_i using Euler's method with $h = \Delta t$

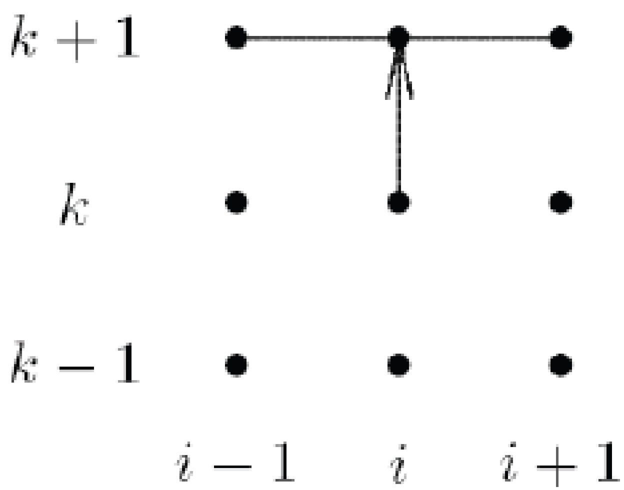
Stability of Finite Differences

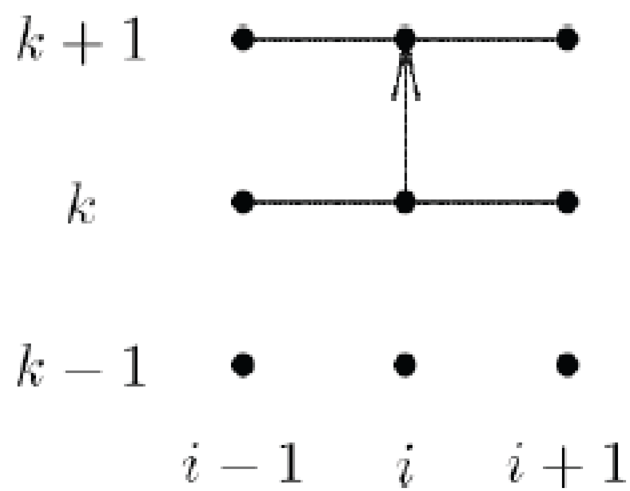
- Rewrite as:

$$y' = \frac{c}{(\Delta x)^2} \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 \end{bmatrix} y = Ay$$

- For forward Euler to be stable, must have $\Delta t <= (\Delta x)^2 / 2c$
- Quite restrictive on Δt!
  - Equivalent approach for advection equation turns out to be unconditionally unstable

Alternative Stencils





- Unconditionally stable with respect to Δt
- (Again, no comment on accuracy)

Lax Equivalence Theorem

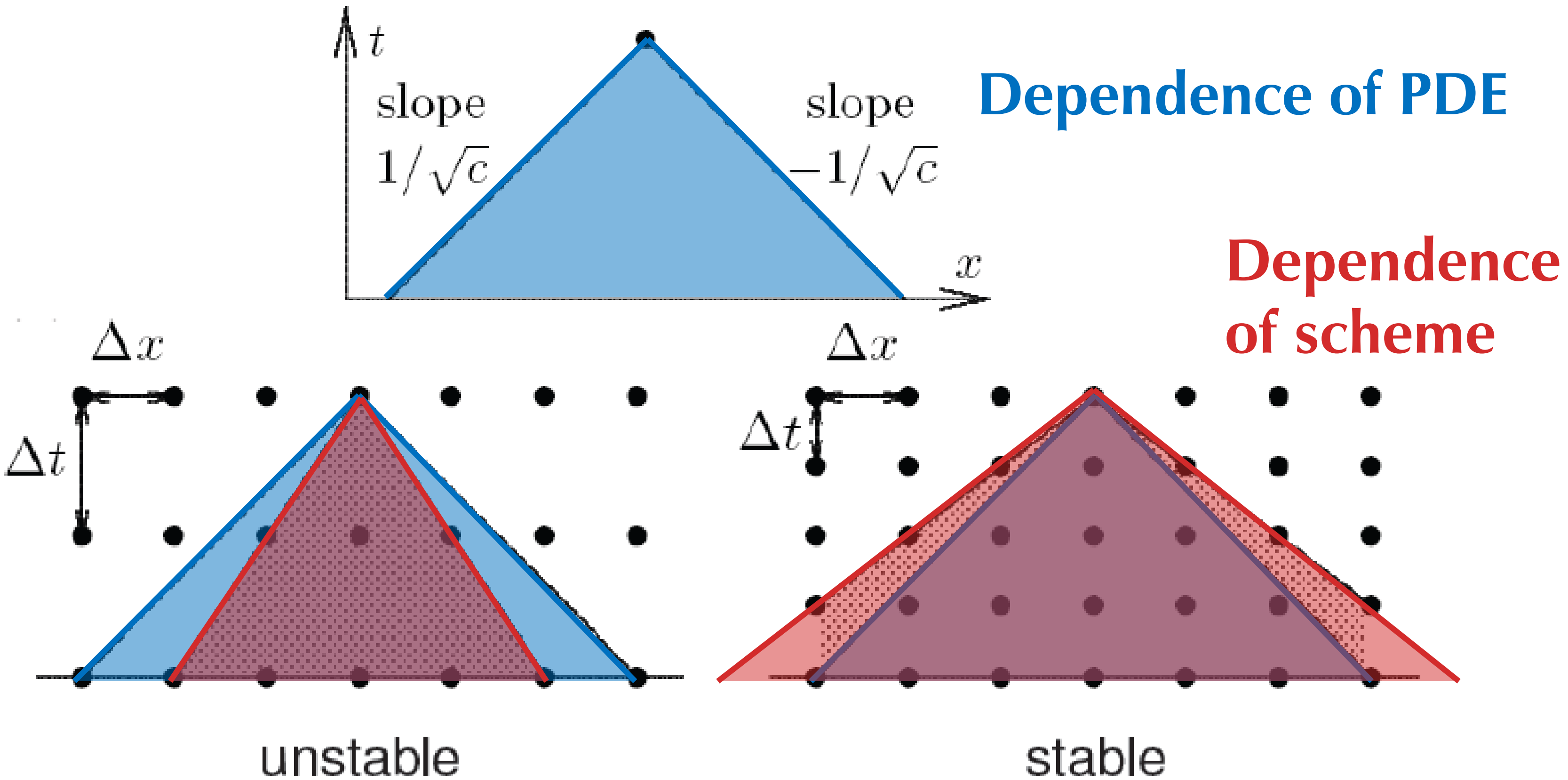
- For a well-posed linear PDE, two necessary and sufficient conditions for finite difference scheme to converge to true solution as Δx and $\Delta t \to 0$:
  - Consistency: local truncation error goes to zero
  - Stability: solution remains bounded
  - Both are required
- Consistency derived from soundness of approximation to derivatives as $\Delta t \to 0$
  - i.e., does numerical method approximate the correct PDE?
- Stability: exact analysis often difficult (but less difficult than showing convergence directly)

Reasoning about PDE Stability

- Matrix method
  - Shown on previous slides
- Domains of dependence

Domains of Dependence

- Courant–Friedrichs–Lewy (CFL) condition: For each mesh point, the **domain of dependence of the PDE** must lie within the **domain of dependence of the finite difference scheme**



Notes on CFL Conditions

- Encapsulated in "CFL Number" or "Courant number" that relates Δt to Δx for a particular equation
- CFL conditions are ***necessary*** but not ***sufficient***
- Can be very restrictive on choice of Δt
- Implicit methods may not require low CFL number for stability, but still may require low number for accuracy

Multigrid Methods

See Heath slides