Finite Difference Approximations For Derivatives
Taylor Series

• Goal: given smooth function $f : \mathbb{R} \rightarrow \mathbb{R}$, find approximate derivatives at some point $x$

• Consider Taylor series expansions around $x$:

\[
\begin{align*}
  f(x + h) &= f(x) + f'(x)h + \frac{f''(x)}{2} h^2 + \frac{f'''(x)}{2} h^3 + \ldots \\
  f(x - h) &= f(x) - f'(x)h + \frac{f''(x)}{2} h^2 - \frac{f'''(x)}{2} h^3 + \ldots
\end{align*}
\]
Forward Difference

• Starting from first equation,

\[ f(x + h) \approx f(x) + f'(x)h + O(h^2) \]

\[ f'(x) \approx \frac{f(x + h) - f(x)}{h} + O(h) \]

• This is the forward-difference approximation to the first derivative: first-order accurate
Backward Difference

- Similarly, starting from second equation,

\[ f(x - h) \approx f(x) - f'(x)h + O(h^2) \]

\[ f'(x) \approx \frac{f(x) - f(x - h)}{h} + O(h) \]

- This is the \textit{backward-difference} approximation to the first derivative: also first-order accurate
Centered Difference

- Subtract the two Taylor-series expansions:

\[ f(x + h) = f(x) + f'(x)h + \frac{f''(x)}{2} h^2 + \frac{f'''(x)}{2} h^3 + \ldots \]

\[ f(x - h) = f(x) - f'(x)h + \frac{f''(x)}{2} h^2 - \frac{f'''(x)}{2} h^3 + \ldots \]

\[ f(x + h) - f(x - h) \approx 2f'(x)h + O(h^3) \]

\[ f'(x) \approx \frac{f(x + h) - f(x - h)}{2h} + O(h^2) \]

- This is the centered-difference approximation to the first derivative: second-order accurate
Second Derivative

• Now add the two Taylor-series expansions:

\[
\begin{align*}
 f(x + h) &= f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(x)}{2}h^3 + \ldots \\
 f(x - h) &= f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(x)}{2}h^3 + \ldots
\end{align*}
\]

\[
\begin{align*}
 f(x + h) + f(x - h) &\approx 2f(x) + f''(x)h^2 + O(h^4) \\
 f''(x) &\approx \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} + O(h^2)
\end{align*}
\]

• This is the centered-difference approximation to the second derivative: second-order accurate