Scientific Computing: An Introductory Survey

Chapter 10 – Boundary Value Problems for Ordinary Differential Equations

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Outline

Boundary Value Problems

- Side conditions prescribing solution or derivative values at specified points are required to make solution of ODE unique
- For initial value problem, all side conditions are specified at single point, say t_0
- For *boundary value problem* (BVP), side conditions are specified at more than one point
- kth order ODE, or equivalent first-order system, requires k side conditions
- For ODEs, side conditions are typically specified at endpoints of interval $[a, b]$, so we have *two-point boundary value problem* with boundary conditions (BC) at a and b.

Existence and Uniqueness

- Unlike IVP, with BVP we cannot begin at initial point and continue solution step by step to nearby points
- Instead, solution is determined everywhere simultaneously, so existence and/or uniqueness may not hold
- For example,

$$u'' = -u, \qquad 0 < t < b$$

with BC

$$u(0) = 0, \qquad u(b) = \beta$$

with b integer multiple of π, has infinitely many solutions if $\beta = 0$, but no solution if $\beta \neq 0$

Existence and Uniqueness, continued

- In general, solvability of BVP

$$\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y}), \qquad a < t < b$$

with BC

$$\boldsymbol{g}(\boldsymbol{y}(a), \boldsymbol{y}(b)) = \boldsymbol{0}$$

depends on solvability of algebraic equation

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}(b; \boldsymbol{x})) = \boldsymbol{0}$$

where $\boldsymbol{y}(t; \boldsymbol{x})$ denotes solution to ODE with initial condition $\boldsymbol{y}(a) = \boldsymbol{x}$ for $\boldsymbol{x} \in \mathbb{R}^n$

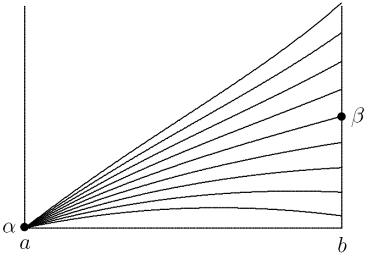
- Solvability of latter system is difficult to establish if $\boldsymbol{g}$ is nonlinear

Numerical Methods for BVPs

- For IVP, initial data supply all information necessary to begin numerical solution method at initial point and step forward from there
- For BVP, we have insufficient information to begin step-by-step numerical method, so numerical methods for solving BVPs are more complicated than those for solving IVPs
- We will consider four types of numerical methods for two-point BVPs
  - Shooting
  - Finite difference
  - Collocation
  - Galerkin

Shooting Method

- In statement of two-point BVP, we are given value of $u(a)$
- If we also knew value of $u'(a)$, then we would have IVP that we could solve by methods discussed previously
- Lacking that information, we try sequence of increasingly accurate guesses until we find value for $u'(a)$ such that when we solve resulting IVP, approximate solution value at $t = b$ matches desired boundary value, $u(b) = \beta$



Shooting Method, continued

- For given γ, value at b of solution $u(b)$ to IVP

$$u'' = f(t, u, u')$$

  with initial conditions

$$u(a) = \alpha, \qquad u'(a) = \gamma$$

  can be considered as function of γ, say $g(\gamma)$

- Then BVP becomes problem of solving equation $g(\gamma) = \beta$

- One-dimensional zero finder can be used to solve this scalar equation

Example: Shooting Method

- Consider two-point BVP for second-order ODE

$$u'' = 6t, \qquad 0 < t < 1$$

with BC

$$u(0) = 0, \qquad u(1) = 1$$

- For each guess for $u'(0)$, we will integrate resulting IVP using classical fourth-order Runge-Kutta method to determine how close we come to hitting desired solution value at $t = 1$
- For simplicity of illustration, we will use step size $h = 0.5$ to integrate IVP from $t = 0$ to $t = 1$ in only two steps
- First, we transform second-order ODE into system of two first-order ODEs

$$\boldsymbol{y}'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} y_2 \\ 6t \end{bmatrix}$$

Example, continued

- We first try guess for initial slope of $y_2(0) = 1$

$$\boldsymbol{y}^{(1)} = \boldsymbol{y}^{(0)} + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1.0 \\ 1.5 \end{bmatrix} + 2\begin{bmatrix} 1.375 \\ 1.500 \end{bmatrix} + \begin{bmatrix} 1.75 \\ 3.00 \end{bmatrix}\right) = \begin{bmatrix} 0.625 \\ 1.750 \end{bmatrix}$$

$$\boldsymbol{y}^{(2)} = \begin{bmatrix} 0.625 \\ 1.750 \end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix} 1.75 \\ 3.00 \end{bmatrix} + 2\begin{bmatrix} 2.5 \\ 4.5 \end{bmatrix} + 2\begin{bmatrix} 2.875 \\ 4.500 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

- So we have hit $y_1(1) = 2$ instead of desired value $y_1(1) = 1$

Example, continued

- We try again, this time with initial slope $y_2(0) = -1$

$$\boldsymbol{y}^{(1)} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -1.0 \\ 1.5 \end{bmatrix} + 2\begin{bmatrix} -0.625 \\ 1.500 \end{bmatrix} + \begin{bmatrix} -0.25 \\ 3.00 \end{bmatrix} \right) = \begin{bmatrix} -0.375 \\ -0.250 \end{bmatrix}$$

$$\boldsymbol{y}^{(2)} = \begin{bmatrix} -0.375 \\ -0.250 \end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix} -0.25 \\ 3.00 \end{bmatrix} + 2\begin{bmatrix} 0.5 \\ 4.5 \end{bmatrix} + 2\begin{bmatrix} 0.875 \\ 4.500 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

- So we have hit $y_1(1) = 0$ instead of desired value $y_1(1) = 1$, but we now have initial slope bracketed between -1 and 1

Example, continued

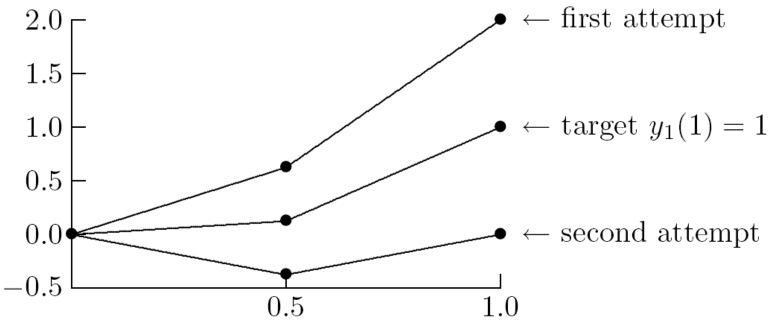
- We omit further iterations necessary to identify correct initial slope, which turns out to be $y_2(0) = 0$

$$\boldsymbol{y}^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0.0 \\ 1.5 \end{bmatrix} + 2\begin{bmatrix} 0.375 \\ 1.500 \end{bmatrix} + \begin{bmatrix} 0.75 \\ 3.00 \end{bmatrix}\right) = \begin{bmatrix} 0.125 \\ 0.750 \end{bmatrix}$$

$$\boldsymbol{y}^{(2)} = \begin{bmatrix} 0.125 \\ 0.750 \end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix} 0.75 \\ 3.00 \end{bmatrix} + 2\begin{bmatrix} 1.5 \\ 4.5 \end{bmatrix} + 2\begin{bmatrix} 1.875 \\ 4.500 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

- So we have indeed hit target solution value $y_1(1) = 1$

Example, continued



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Finite Difference Method

- *Finite difference method* converts BVP into system of algebraic equations by replacing all derivatives with finite difference approximations
- For example, to solve two-point BVP

$$u'' = f(t, u, u'), \qquad a < t < b$$

with BC

$$u(a) = \alpha, \qquad u(b) = \beta$$

we introduce mesh points $t_i = a + ih$, $i = 0, 1, \ldots, n+1$, where $h = (b-a)/(n+1)$

- We already have $y_0 = u(a) = \alpha$ and $y_{n+1} = u(b) = \beta$ from BC, and we seek approximate solution value $y_i \approx u(t_i)$ at each interior mesh point t_i, $i = 1, \ldots, n$

Finite Difference Method, continued

- We replace derivatives by finite difference approximations such as

$$u'(t_i) \approx \frac{y_{i+1} - y_{i-1}}{2h}$$

$$u''(t_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

- This yields system of equations

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = f\left(t_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right)$$

  to be solved for unknowns y_i, $i = 1, \ldots, n$

- System of equations may be linear or nonlinear, depending on whether f is linear or nonlinear

Finite Difference Method, continued

- For these particular finite difference formulas, system to be solved is tridiagonal, which saves on both work and storage compared to general system of equations
- This is generally true of finite difference methods: they yield sparse systems because each equation involves few variables

Example: Finite Difference Method

- Consider again two-point BVP

$$u'' = 6t, \qquad 0 < t < 1$$

  with BC

$$u(0) = 0, \qquad u(1) = 1$$

- To keep computation to minimum, we compute approximate solution at one interior mesh point, $t = 0.5$, in interval $[0, 1]$
- Including boundary points, we have three mesh points, $t_0 = 0$, $t_1 = 0.5$, and $t_2 = 1$
- From BC, we know that $y_0 = u(t_0) = 0$ and $y_2 = u(t_2) = 1$, and we seek approximate solution $y_1 \approx u(t_1)$

Example, continued

- Replacing derivatives by standard finite difference approximations at t_1 gives equation

$$\frac{y_2 - 2y_1 + y_0}{h^2} = f\left(t_1, y_1, \frac{y_2 - y_0}{2h}\right)$$

- Substituting boundary data, mesh size, and right hand side for this example we obtain

$$\frac{1 - 2y_1 + 0}{(0.5)^2} = 6t_1$$

or

$$4 - 8y_1 = 6(0.5) = 3$$

so that

$$y(0.5) \approx y_1 = 1/8 = 0.125$$

Example, continued

- In a practical problem, much smaller step size and many more mesh points would be required to achieve acceptable accuracy
- We would therefore obtain *system* of equations to solve for approximate solution values at mesh points, rather than single equation as in this example

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Collocation Method

- *Collocation method* approximates solution to BVP by finite linear combination of basis functions
- For two-point BVP

$$u'' = f(t, u, u'), \qquad a < t < b$$

with BC

$$u(a) = \alpha, \qquad u(b) = \beta$$

we seek approximate solution of form

$$u(t) \approx v(t, \boldsymbol{x}) = \sum_{i=1}^{n} x_i \phi_i(t)$$

where ϕ_i are basis functions defined on $[a, b]$ and $\boldsymbol{x}$ is n-vector of parameters to be determined

Collocation Method

- Popular choices of basis functions include polynomials, B-splines, and trigonometric functions
- Basis functions with global support, such as polynomials or trigonometric functions, yield *spectral method*
- Basis functions with highly localized support, such as B-splines, yield *finite element method*

Collocation Method, continued

- To determine vector of parameters $\boldsymbol{x}$, define set of n *collocation points*, $a = t_1 < \cdots < t_n = b$, at which approximate solution $v(t, \boldsymbol{x})$ is forced to satisfy ODE and boundary conditions
- Common choices of collocation points include equally-spaced points or Chebyshev points
- Suitably smooth basis functions can be differentiated analytically, so that approximate solution and its derivatives can be substituted into ODE and BC to obtain system of algebraic equations for unknown parameters $\boldsymbol{x}$

Example: Collocation Method

- Consider again two-point BVP

$$u'' = 6t, \qquad 0 < t < 1,$$

with BC

$$u(0) = 0, \qquad u(1) = 1$$

- To keep computation to minimum, we use one interior collocation point, $t = 0.5$
- Including boundary points, we have three collocation points, $t_0 = 0$, $t_1 = 0.5$, and $t_2 = 1$, so we will be able to determine three parameters
- As basis functions we use first three monomials, so approximate solution has form

$$v(t, \boldsymbol{x}) = x_1 + x_2 t + x_3 t^2$$

Example, continued

- Derivatives of approximate solution function with respect to t are given by

$$v'(t, \boldsymbol{x}) = x_2 + 2x_3 t, \qquad v''(t, \boldsymbol{x}) = 2x_3$$

- Requiring ODE to be satisfied at interior collocation point $t_2 = 0.5$ gives equation

$$v''(t_2, \boldsymbol{x}) = f(t_2, v(t_2, \boldsymbol{x}), v'(t_2, \boldsymbol{x}))$$

or

$$2x_3 = 6t_2 = 6(0.5) = 3$$

- Boundary condition at $t_1 = 0$ gives equation

$$x_1 + x_2 t_1 + x_3 t_1^2 = x_1 = 0$$

- Boundary condition at $t_3 = 1$ gives equation

$$x_1 + x_2 t_3 + x_3 t_3^2 = x_1 + x_2 + x_3 = 1$$

Example, continued

- Solving this system of three equations in three unknowns gives

$$x_1 = 0, \qquad x_2 = -0.5, \qquad x_3 = 1.5$$

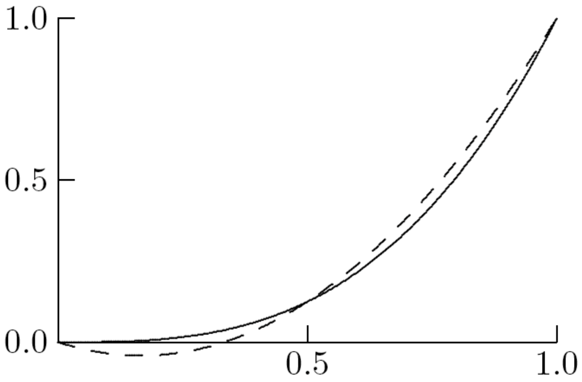
so approximate solution function is quadratic polynomial

$$u(t) \approx v(t, \boldsymbol{x}) = -0.5t + 1.5t^2$$

- At interior collocation point, $t_2 = 0.5$, we have approximate solution value

$$u(0.5) \approx v(0.5, \boldsymbol{x}) = 0.125$$

Example, continued



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