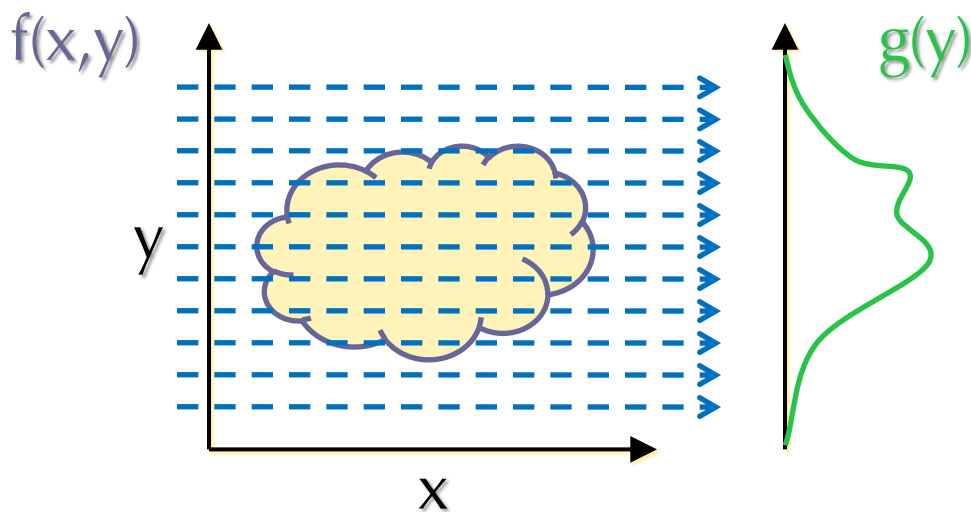


Monte Carlo Integration

COS 323

Integration in d Dimensions?

- One option: nested 1-D integration



$$\iint f(x, y) dx dy = \int g(y) dy$$

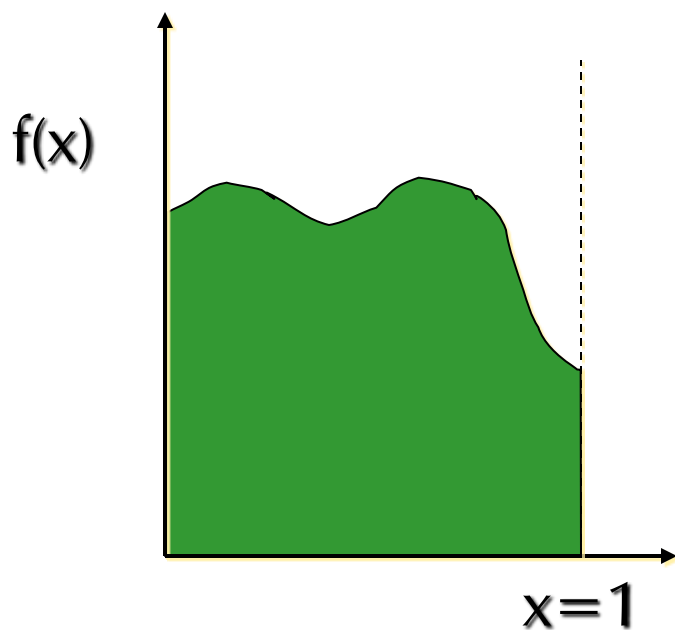
Evaluate the latter numerically, but each “sample” of $g(y)$ is itself a 1-D integral, done numerically

Integration in d Dimensions?

- Midpoint rule in d dimensions?
 - In 1D: $(b-a)/h$ points
 - In 2D: $(b-a)/h^2$ points
 - In general: $O(1/h^d)$ points
- Required # of points ***grows exponentially with dimension***, for a fixed order of method
 - “Curse of dimensionality”
- Other problems, e.g. non-rectangular domains

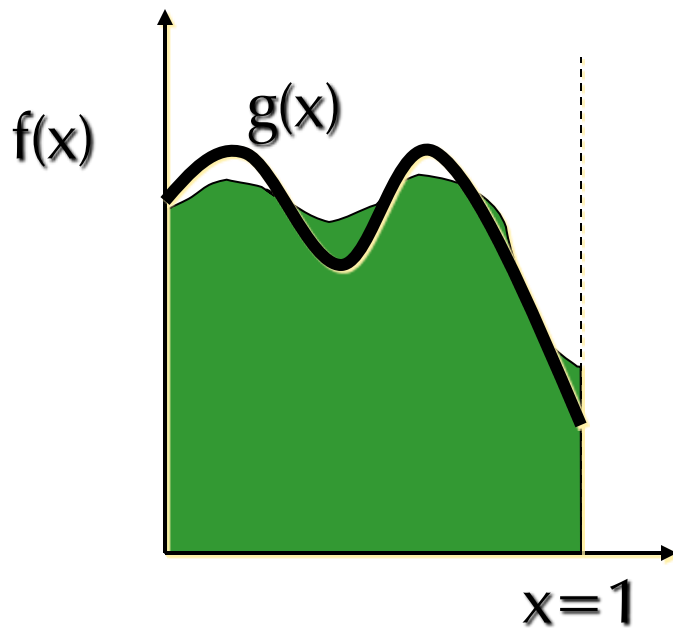
Rethinking Integration in 1D

$$\int_0^1 f(x) dx = ?$$



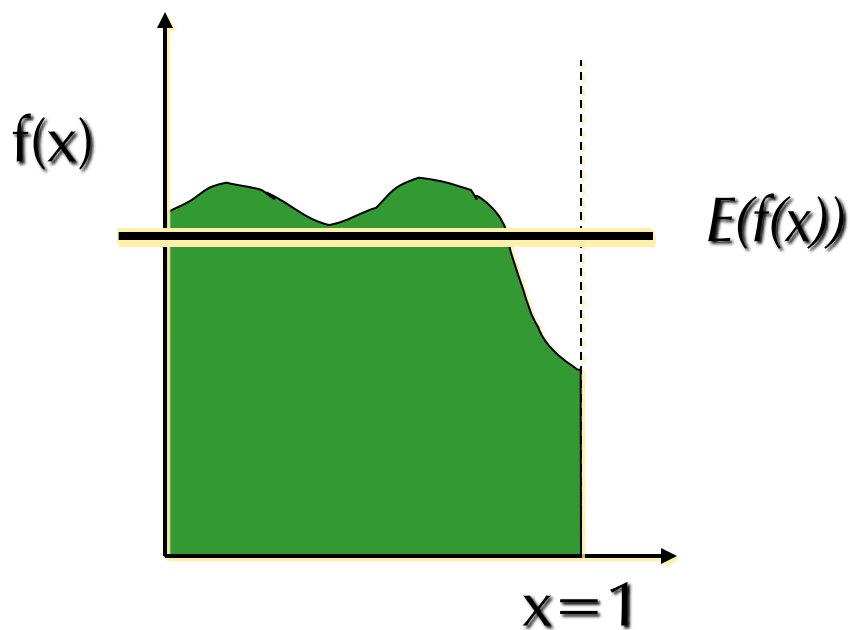
We Can Approximate...

$$\int_0^1 f(x) dx = \int_0^1 g(x) dx$$



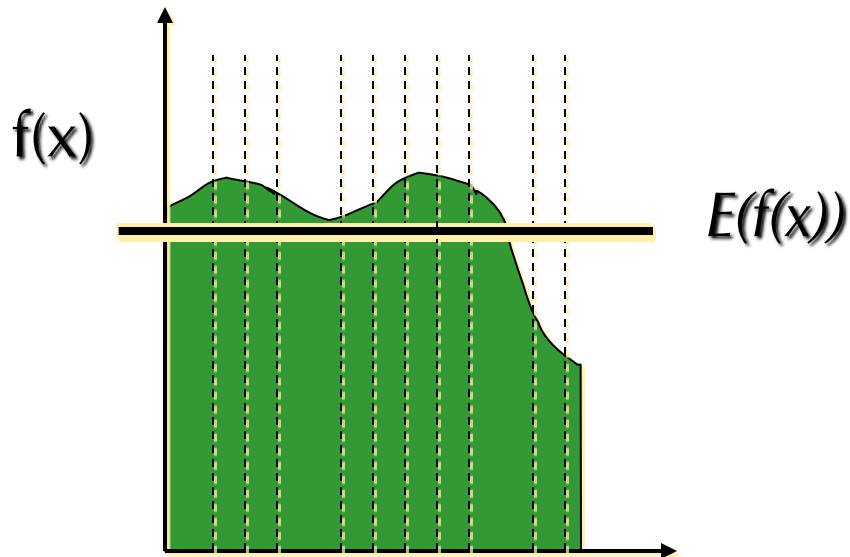
Or We Can Average

$$\int_0^1 f(x) dx = E(f(x))$$



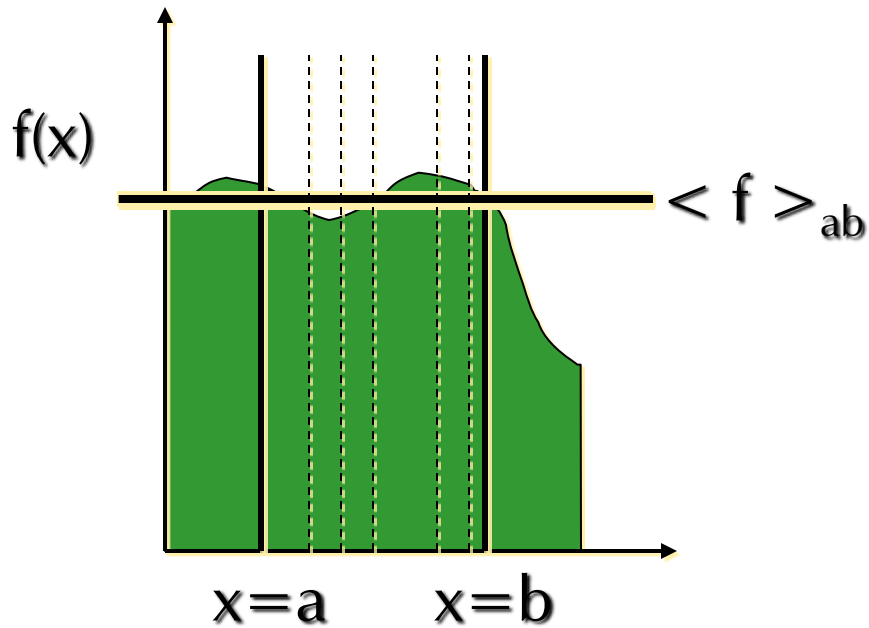
Estimating the Average

$$\int_0^1 f(x) dx \cong \frac{1}{N} \sum_{i=1}^N f(x_i)$$



Other Domains

$$\int_a^b f(x) dx \cong \frac{b-a}{N} \sum_{i=1}^N f(x_i)$$



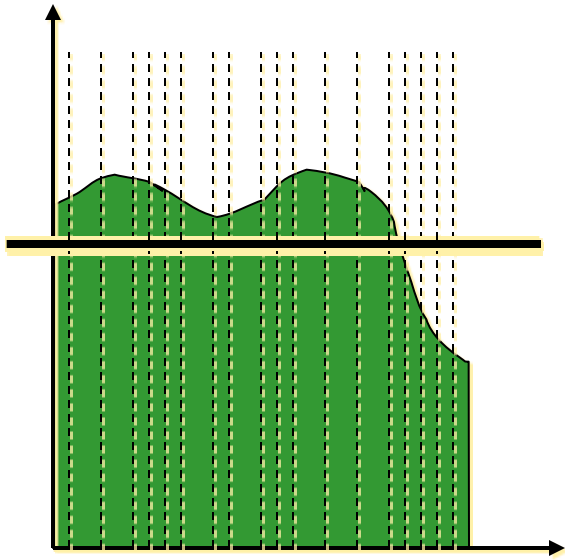
“Monte Carlo” Integration

- No “exponential explosion” in required number of samples with increase in dimension
- (Some) resistance to badly-behaved functions



Le Grand Casino de Monte-Carlo

Variance



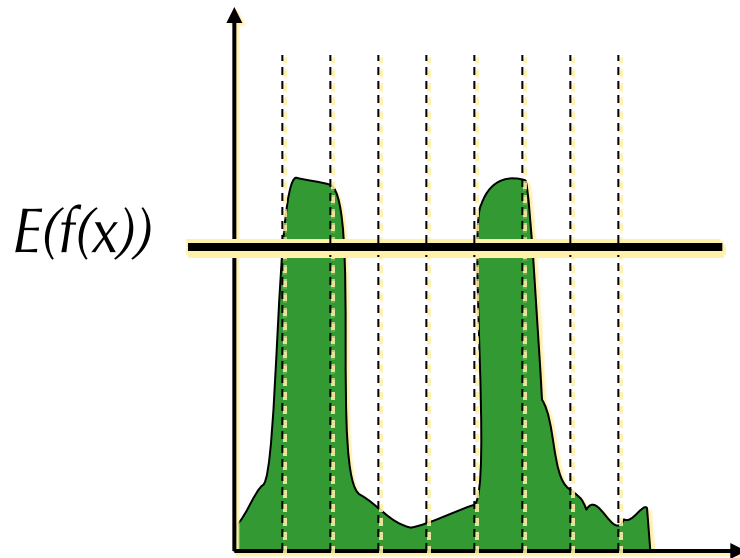
$$\int_a^b f(x) dx \cong \frac{b-a}{N} \sum_{i=1}^N f(x_i)$$
$$\text{Var}\left[\frac{b-a}{N} \sum_{i=1}^N f(x_i)\right] = \left(\frac{b-a}{N}\right)^2 \sum_{i=1}^N \text{Var}[f(x_i)]$$
$$= \frac{(b-a)^2}{N} \text{Var}[f(x_i)]$$

* with a correction of $\sqrt{\frac{N}{N-1}}$
(consult a statistician for details)

Variance decreases as $1/N$
Error of E decreases as $1/\text{sqrt}(N)$

Variance

- Problem: variance decreases with $1/N$
 - Increasing # samples removes noise slowly

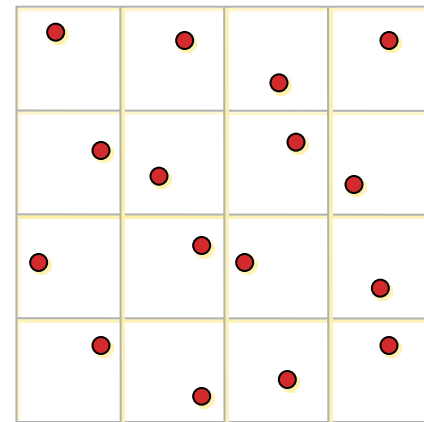
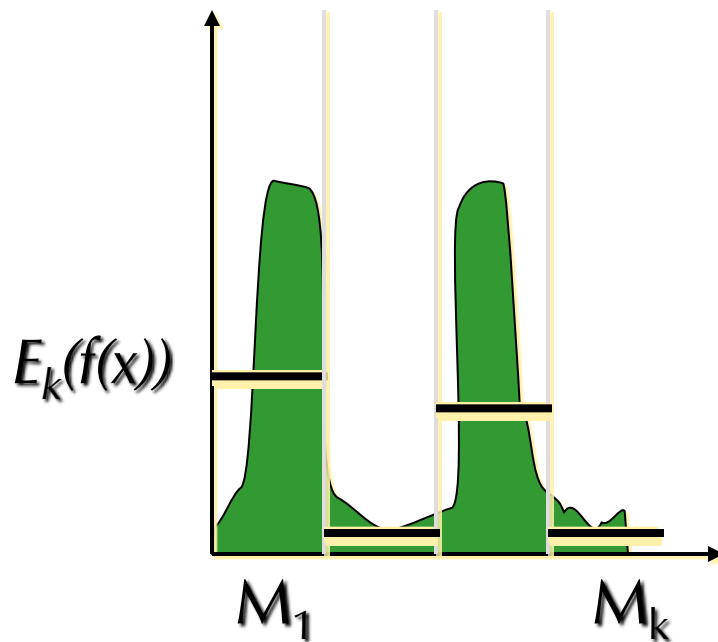


Variance Reduction Techniques

- Problem: variance decreases with $1/N$
 - Increasing # samples removes noise slowly
- Variance reduction:
 - Stratified sampling
 - Importance sampling

Stratified Sampling

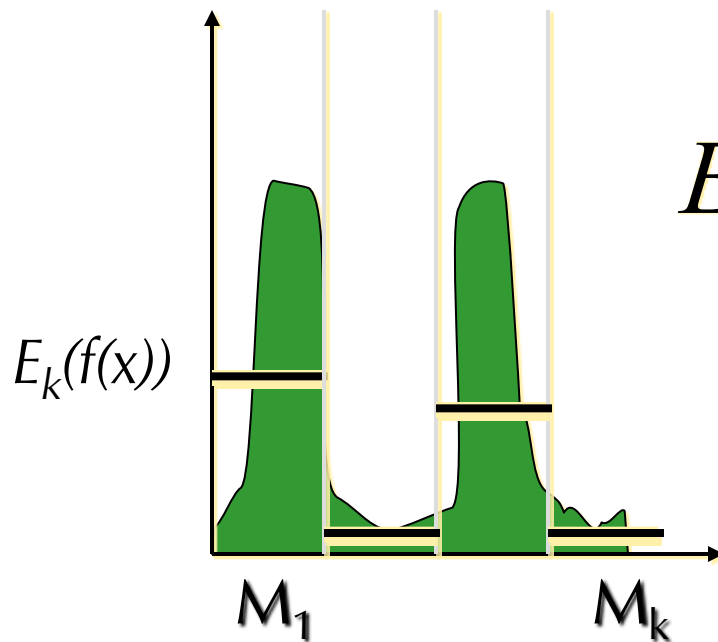
- Estimate subdomains separately



Can do this recursively!

Stratified Sampling

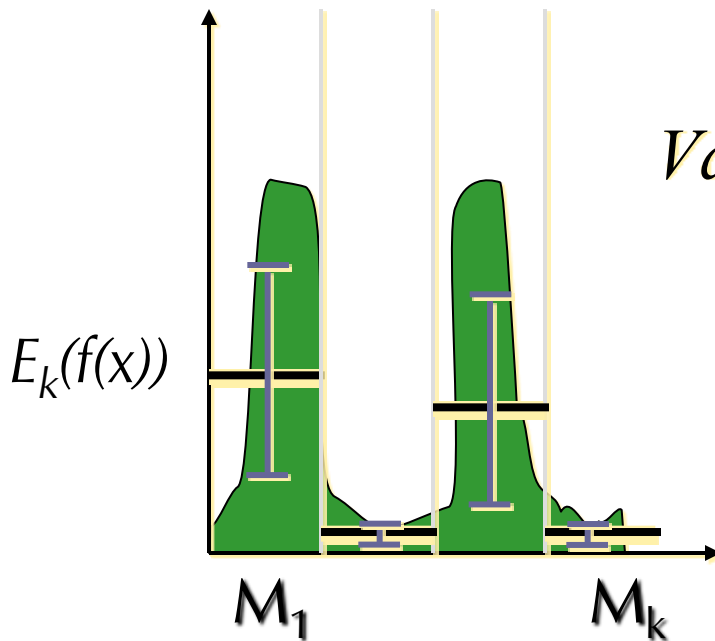
- This is still unbiased



$$E = \sum_{j=1}^k \frac{\text{vol}(M_j)}{N_j} \sum_{n=1}^{N_j} f(x_{jn})$$

Stratified Sampling

- Less overall variance if less variance in subdomains

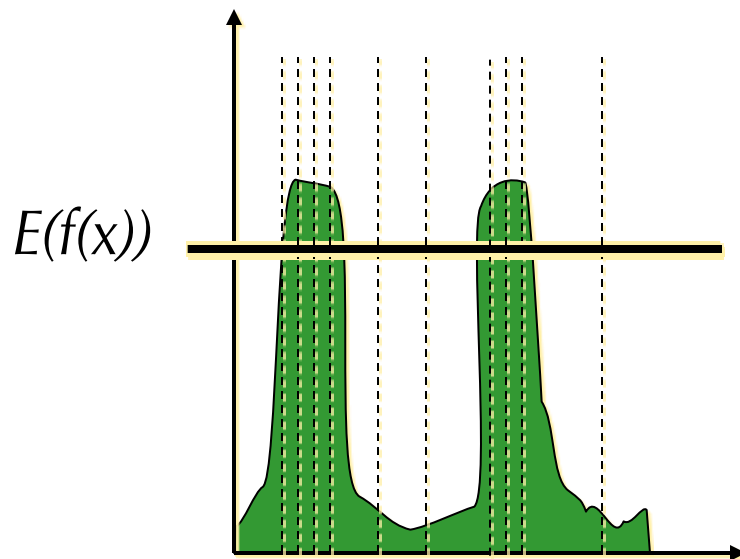


$$Var[E] = \sum_{j=1}^k \frac{vol(M_j)^2}{N_j} Var[f(x)]|_{M_j}$$

Total variance minimized when number of points in each subvolume M_j proportional to error in M_j .

Importance Sampling

- Put more samples where $f(x)$ is bigger



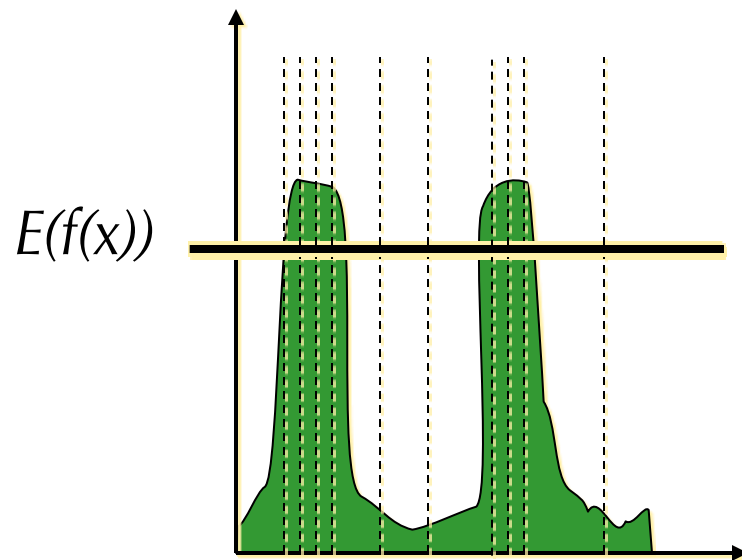
$$\int_{\Omega} f(x) dx = \frac{1}{N} \sum_{i=1}^N Y_i$$

$$\text{where } Y_i = \frac{f(x_i)}{p(x_i)}$$

and x_i drawn from $P(x)$

Importance Sampling

- This is still unbiased



$$E[Y_i] = \int_{\Omega} Y(x) p(x) dx$$

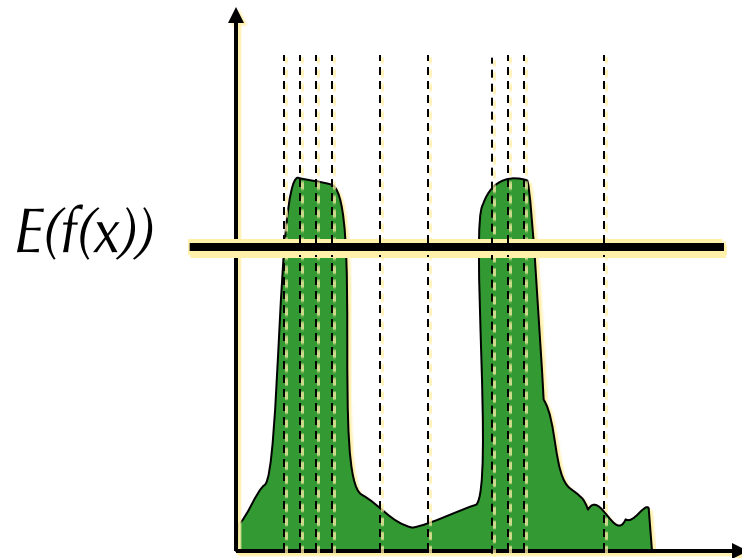
$$= \int_{\Omega} \frac{f(x)}{p(x)} p(x) dx$$

$$= \int_{\Omega} f(x) dx$$

for all N

Importance Sampling

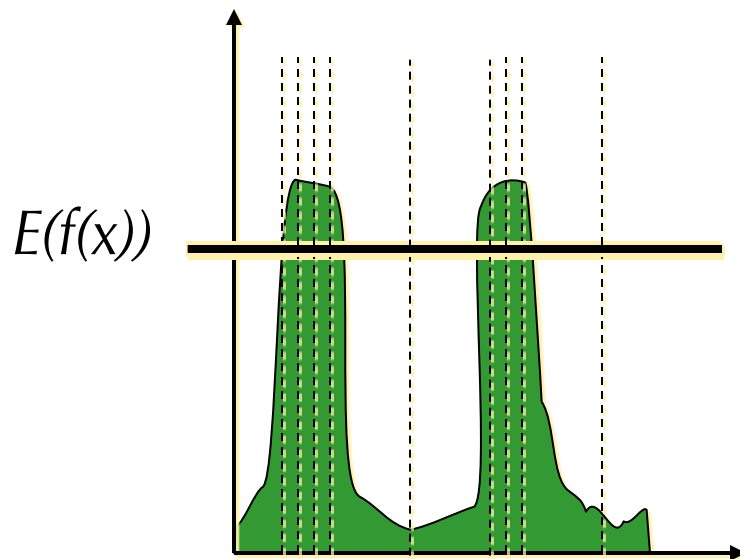
- Variance depends on choice of $p(x)$:



$$Var(E) = \frac{1}{N} \sum_{n=1}^N \left(\frac{f(x_n)}{p(x_n)} \right)^2 - E^2$$

Importance Sampling

- Zero variance if $p(x) \sim f(x)$



$$p(x) = cf(x)$$

$$Y_i = \frac{f(x_i)}{p(x_i)} = \frac{1}{c}$$

$$\text{Var}(Y) = 0$$

Less variance with better
importance sampling

Random number generation

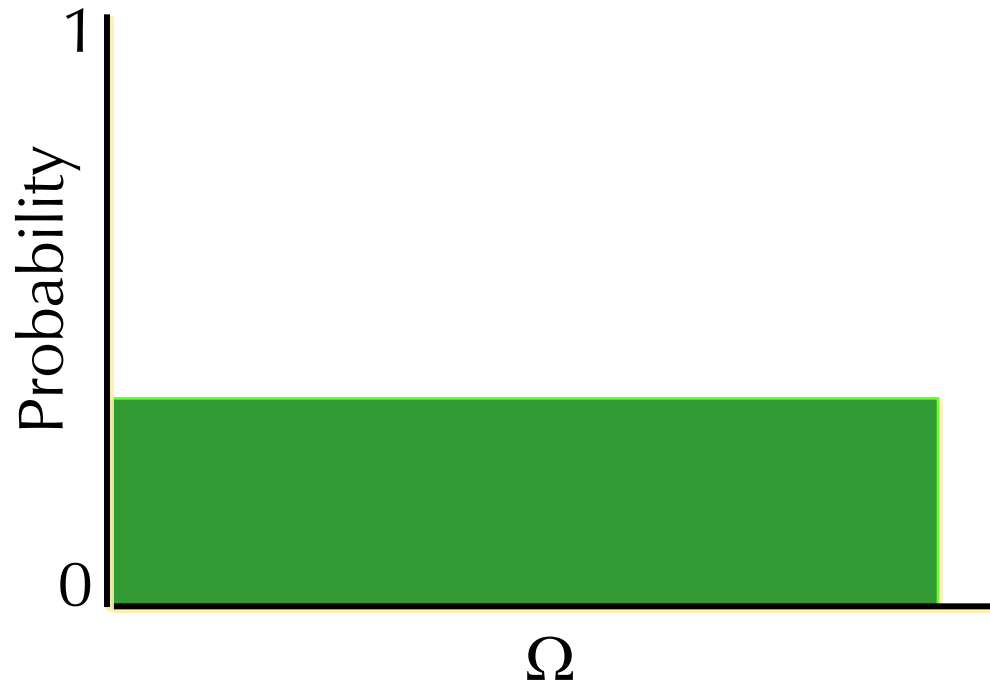
True random numbers

- <http://www.random.org/>

10101111 00101011 10111000 11110110 10101010 00110001 01100011 00010001
00000011 00000010 00111111 00010011 00000101 01001100 10000110 11100010
10010100 10000101 10000011 00000100 00111011 10111000 00110000 11001010
11011101 11101111 00100010 10101011 00100110 10101111 00001011 10110100
00011100 00001111 11001001 11001100 01111101 10000100 10111000 01101011
01101011 01111101 11001010 11101110 11101110 00100010 10110100 01001000
11010111 11011011 11100100 01010010 10111101 01011010 01001110 01110000
00100010 11000111 01010000 10110011 01001011 00110001 01011100 10001111
11111000 10101011 01011011 01010000 01101111 00011001 00000011 00110000
10000001 00000110 11010011 00011110 11101101 00000011 00100110 01010011
11010111 10010001 10000111 01010010 01101010 00100101 10011111 01000111
10101001 01100001 01010011 01001000 11010110 01111110 11010011 01110110
00000001 01001110 00011001 00111001

Generating Random Points

- Uniform distribution:
 - Use pseudorandom number generator



Pseudorandom Numbers

- Deterministic, but have statistical properties resembling true random numbers
- Common approach: each successive pseudorandom number is function of previous

Desirable properties

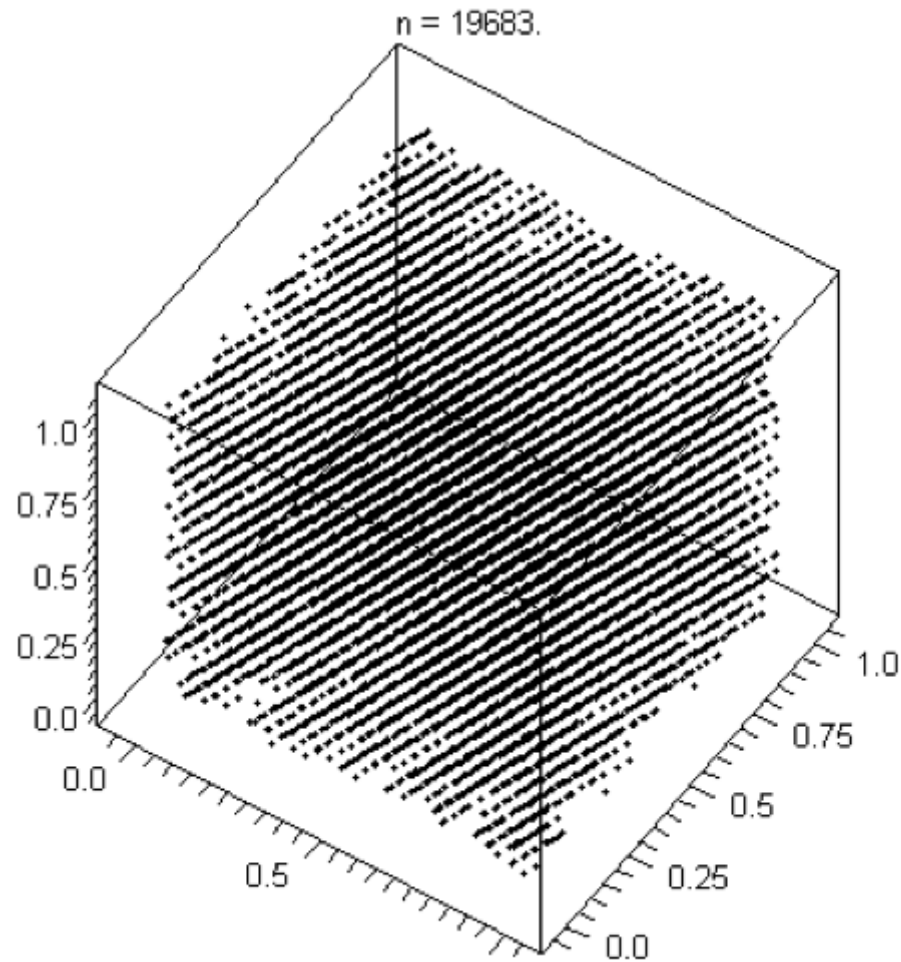
- Random pattern: Passes statistical tests (e.g., can use chi-squared)
- Long period: As long as possible without repeating
- Efficiency
- Repeatability: Produce same sequence if started with same initial conditions (for debugging!)
- Portability

Linear Congruential Methods

$$x_{n+1} = (ax_n + b) \bmod c$$

- Choose constants carefully, e.g.
a = 1664525
b = 1013904223
c = 2^{32}
- Results in integer in $[0, c)$
- Simple, efficient, but often unsuitable for MC:
e.g. exhibit serial correlations

Problem with LCGs



No higher resolution available.

Lagged Fibonacci Generators

- Takes form $x_n = (x_{n-j} \star x_{n-k}) \bmod m$, where operation $\star$ is addition, subtraction, or XOR
- Standard choices of (j, k) : e.g., $(7, 10)$, $(5, 17)$, $(6, 31)$, $(24, 55)$, $(31, 63)$ with $m = 2^{32}$
- Proper initialization is important and hard
- Built-in correlation!
- Not totally understood in theory (need statistical tests to evaluate)

Seeds

- Why?
- Approaches:
 - Ask the user (for debugging)
 - Time of day
 - True random noise: from radio turned to static, or thermal noise in a resistor, or...

Seeds

- Lava lamps!

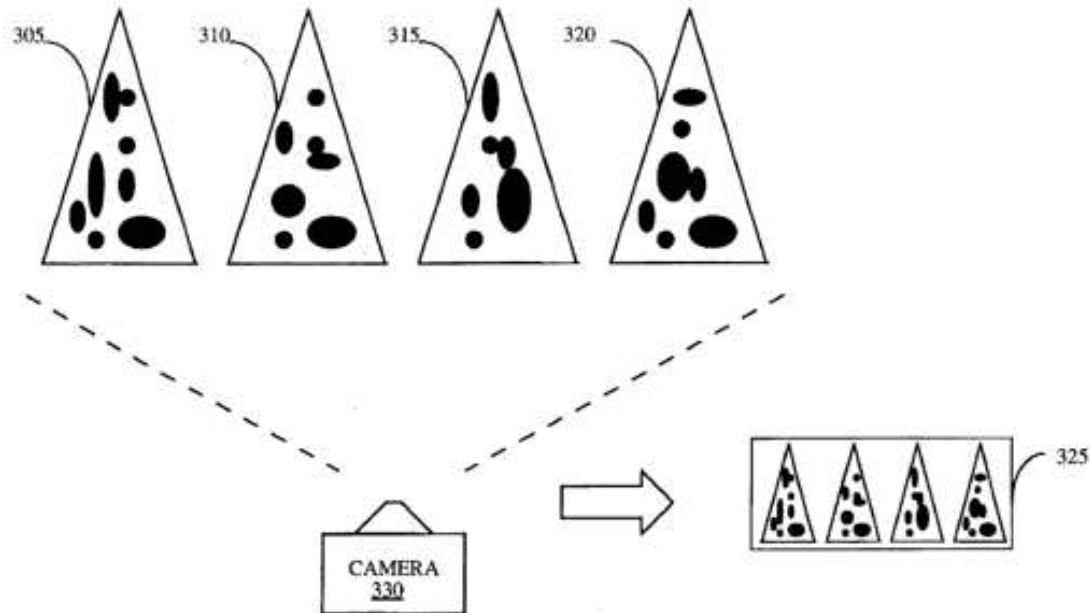


FIG. 3

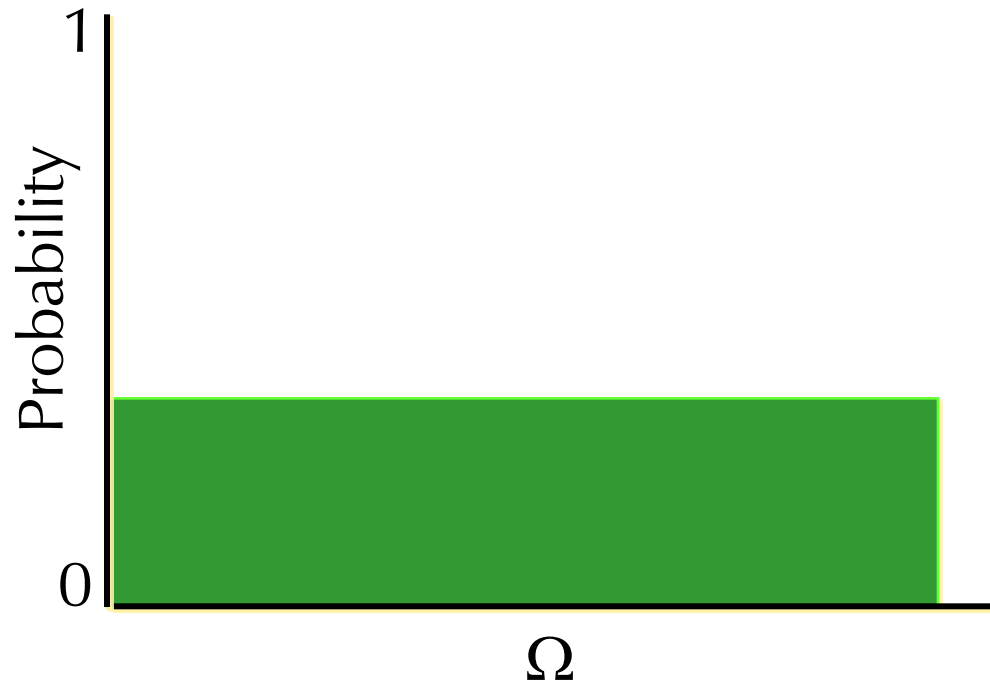
- http://www.google.com/patents/about/5732138_Method_for_seeding_a_pseudo_rand.html?id=ou0gAAAAEBAJ

Pseudorandom Numbers

- Most methods provide integers in range $[0..c)$
- To get floating-point numbers in $[0..1)$, divide integer numbers by c
- To get integers in range $[u..v]$, divide by $c/(v-u+1)$, truncate, and add u
 - Better statistics than using modulo $(v-u+1)$
 - Only works if u and v small compared to c

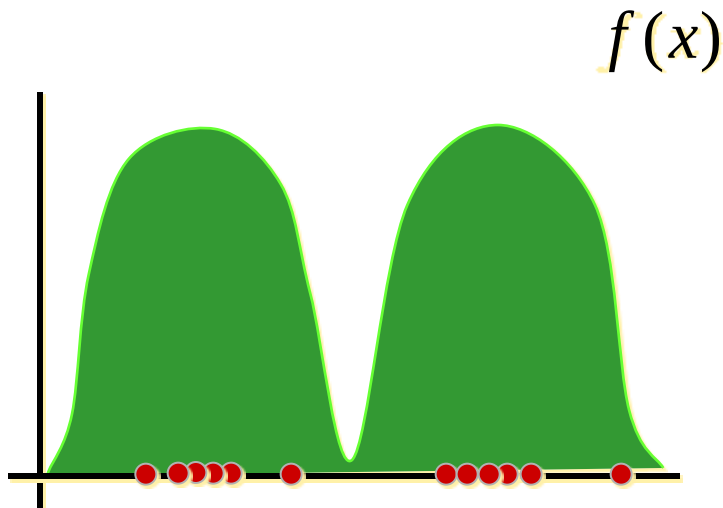
Generating Random Points

- Uniform distribution:
 - Use pseudorandom number generator



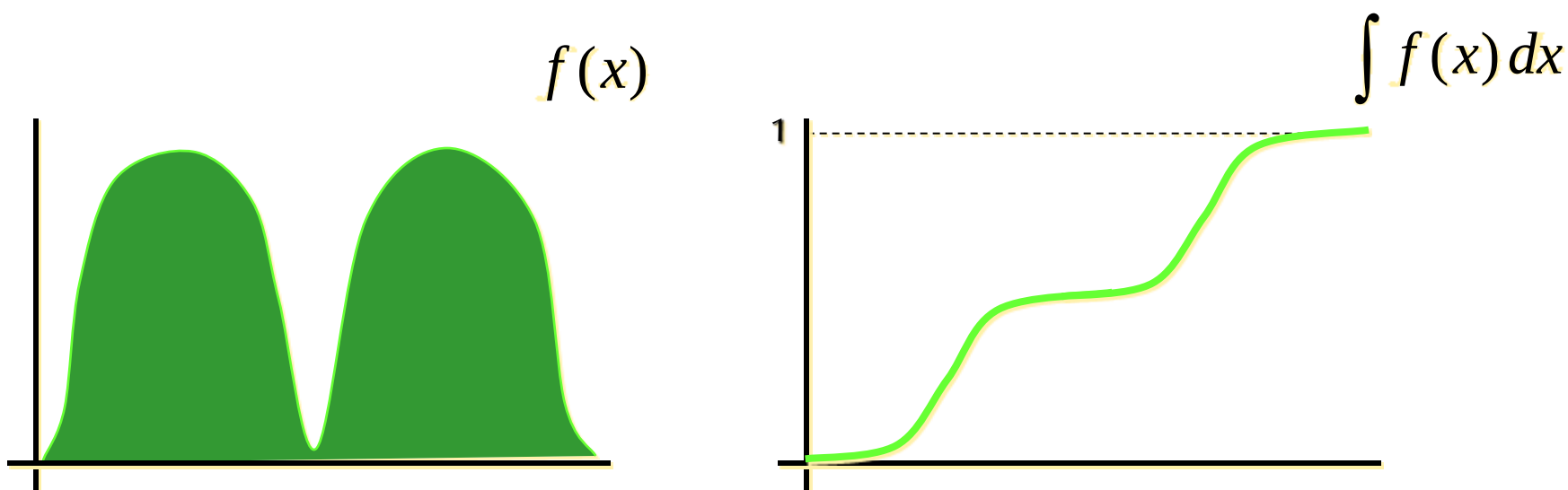
Sampling from a non-uniform distribution

- Specific probability distribution:
 - Function inversion
 - Rejection



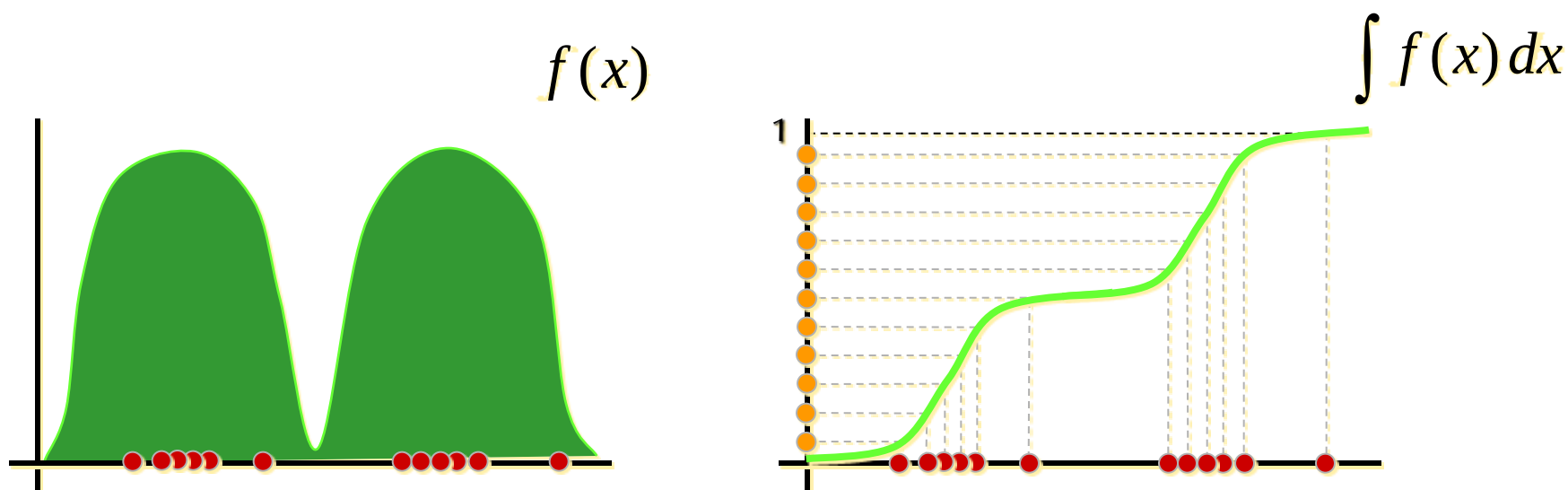
Sampling from a non-uniform distribution

- “Inversion method”
 - Integrate $f(x)$: Cumulative Distribution Function



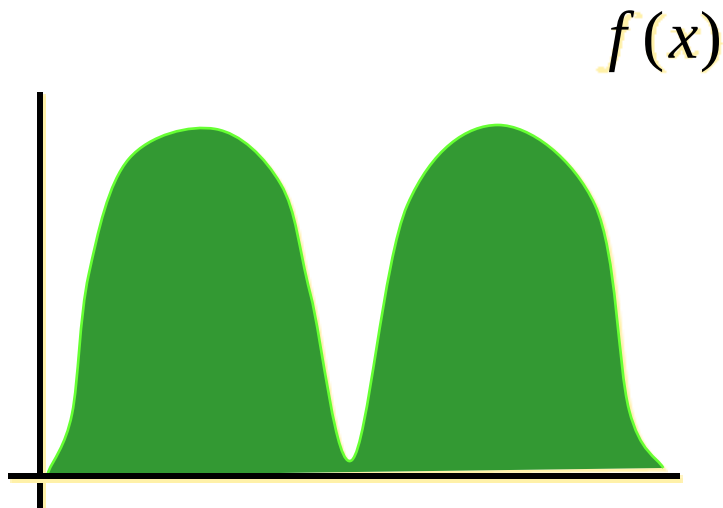
Sampling from a non-uniform distribution

- “Inversion method”
 - Integrate $f(x)$: Cumulative Distribution Function
 - Invert CDF, apply to uniform random variable



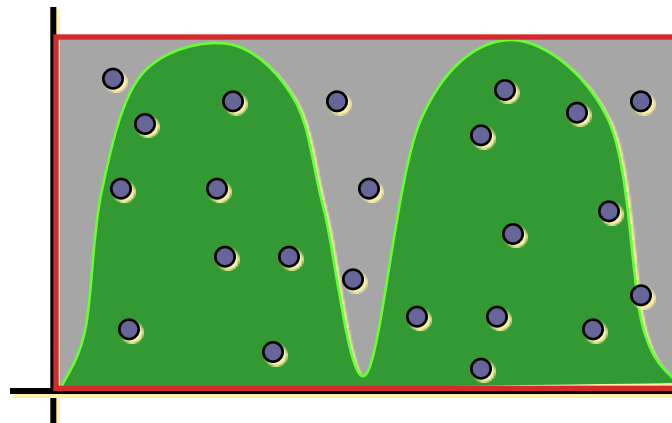
Sampling from a non-uniform distribution

- Specific probability distribution:
 - Function inversion
 - Rejection



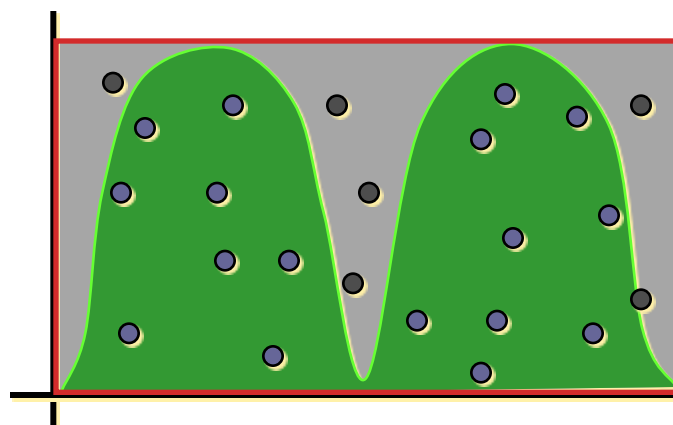
Sampling from a non-uniform distribution

- “Rejection method”
 - Generate random (x,y) pairs,
y between 0 and $\max(f(x))$



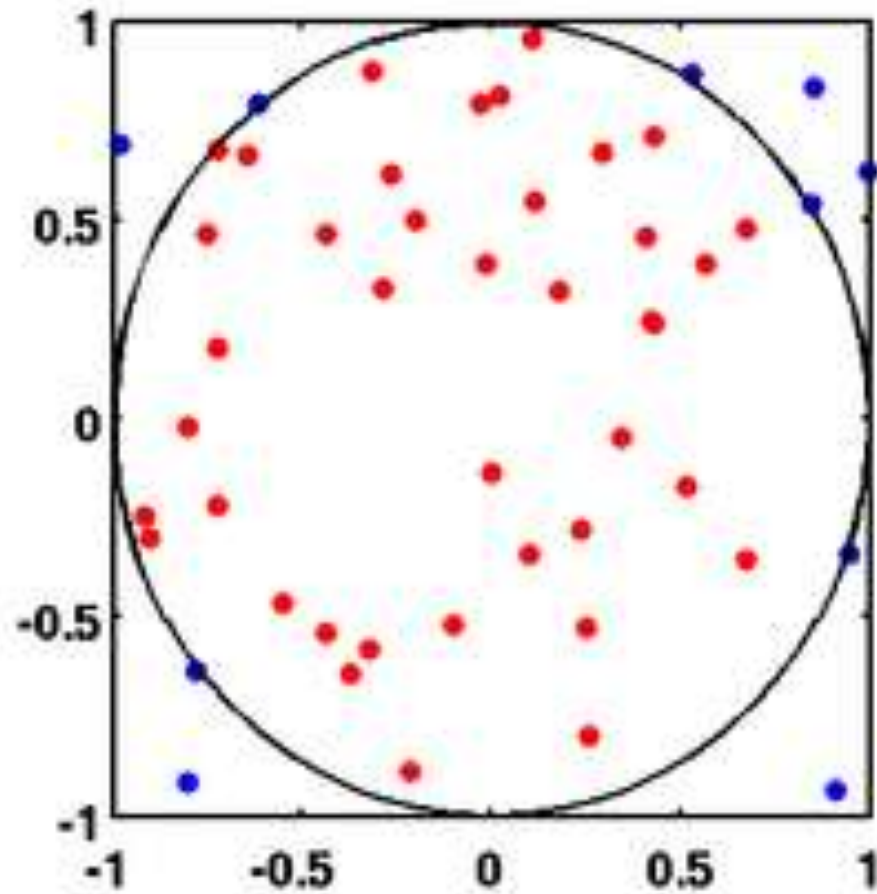
Sampling from a non-uniform distribution

- “Rejection method”
 - Generate random (x,y) pairs,
y between 0 and $\max(f(x))$
 - Keep only samples where $y < f(x)$

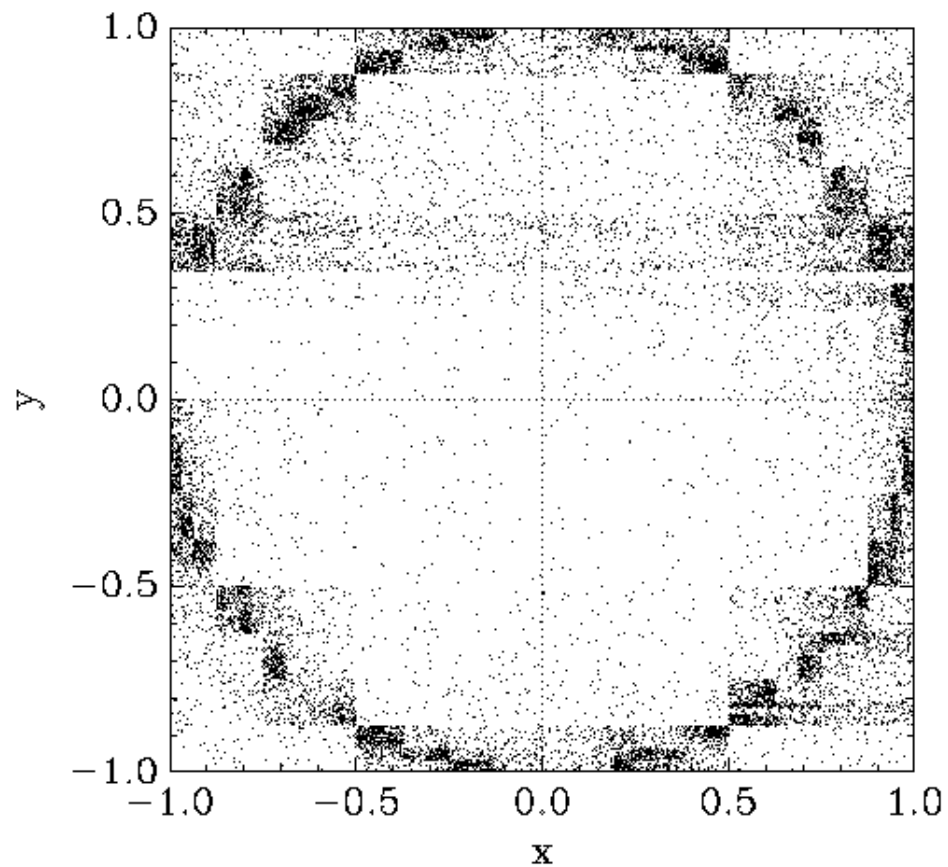


Doesn't require cdf: Can use directly for importance sampling.

Example: Computing pi

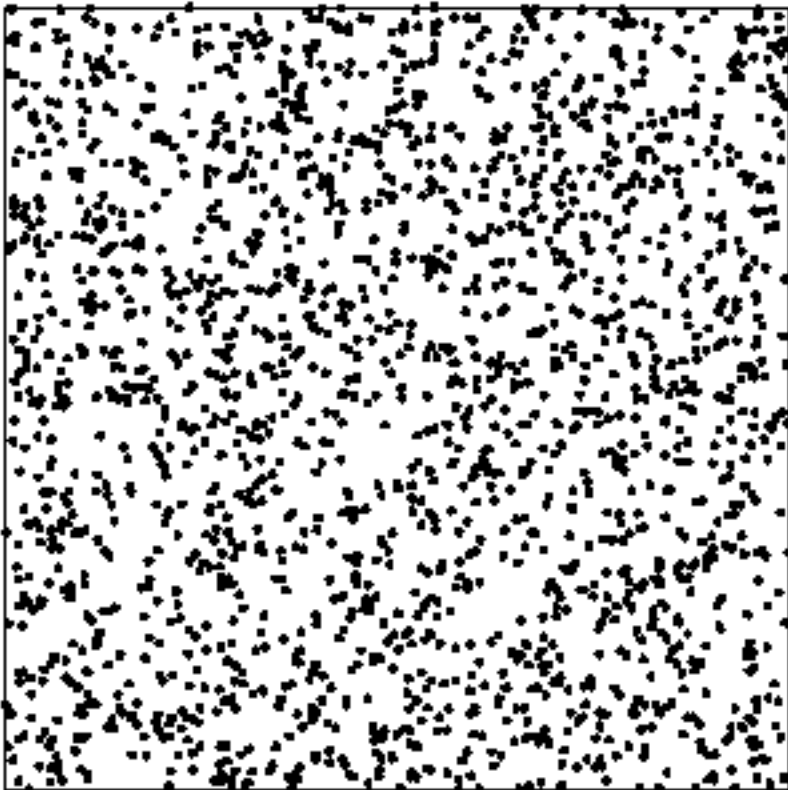


With Stratified Sampling

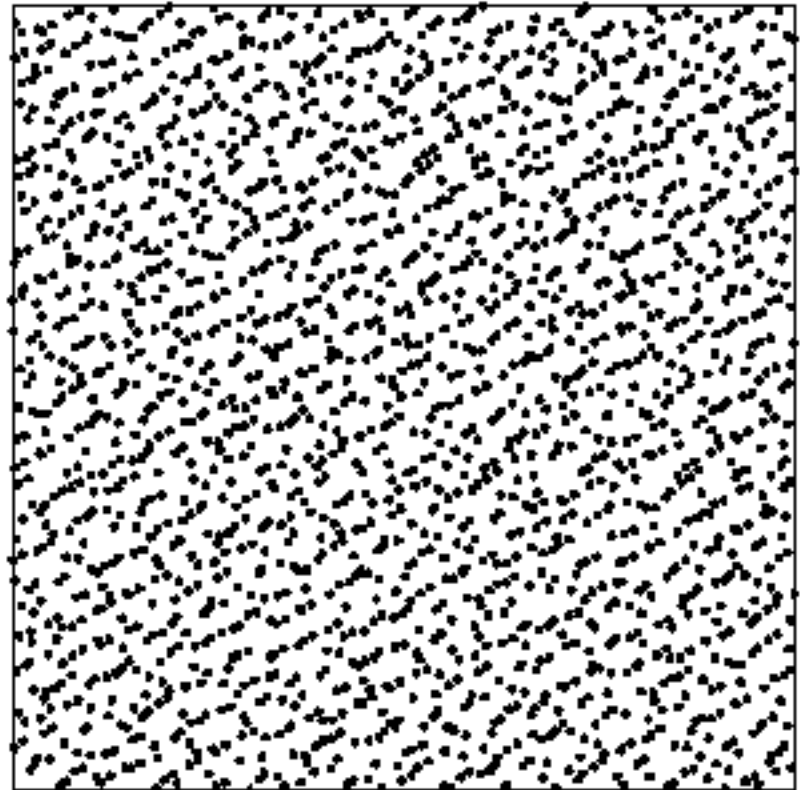


Quasi-Random Sampling

Random



Quasi-Random

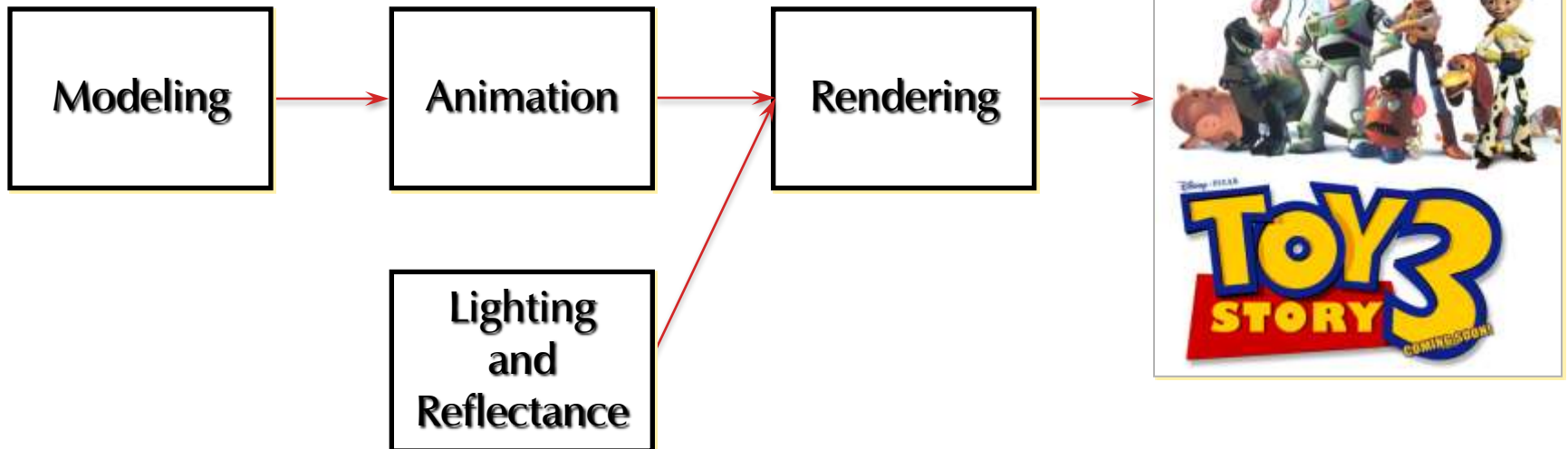


Monte Carlo in Computer Graphics

or, Solving Integral Equations
for Fun and Profit

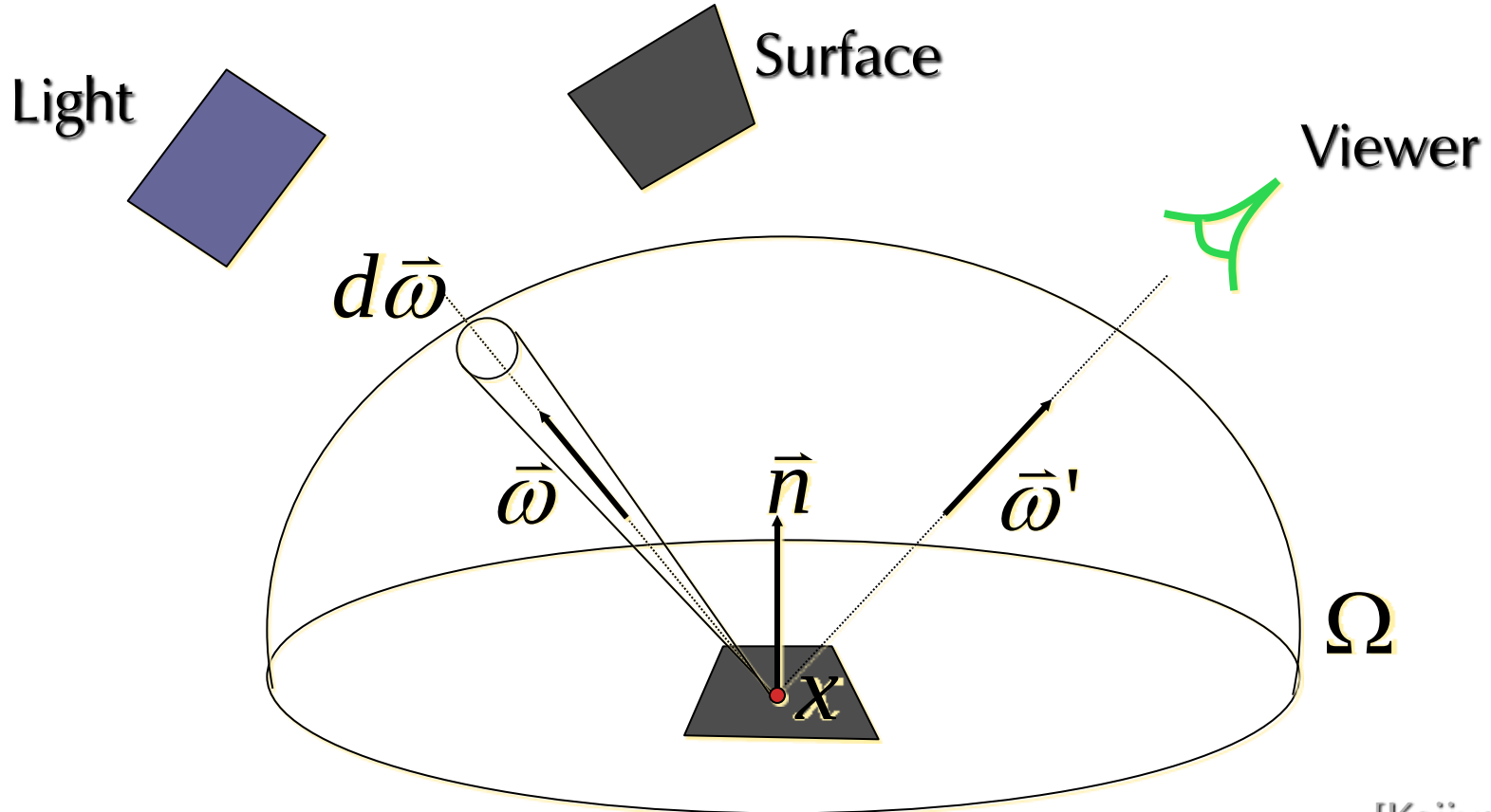
or, Ugly Equations, Pretty Pictures

Computer Graphics Pipeline



Rendering Equation

$$L_o(x, \bar{\omega}') = L_e(x, \bar{\omega}') + \int_{\Omega} L_i(x, \bar{\omega}) f_r(x, \bar{\omega}, \bar{\omega}') (\bar{\omega} \cdot \bar{n}) d\bar{\omega}$$



Rendering Equation

$$L_o(x, \vec{\omega}') = L_e(x, \vec{\omega}') + \int_{\Omega} L_i(x, \vec{\omega}) f_r(x, \vec{\omega}, \vec{\omega}') (\vec{\omega} \cdot \vec{n}) d\vec{\omega}$$

- This is an *integral equation*
- Hard to solve!
 - Can't solve this in closed form
 - Simulate complex phenomena



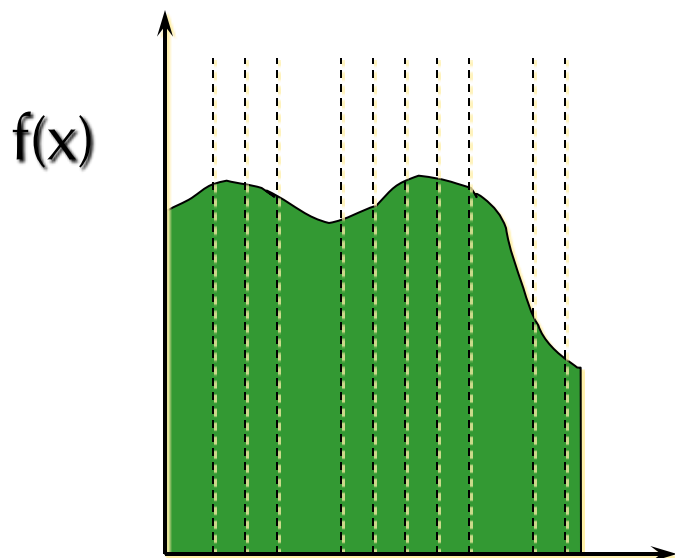
Rendering Equation

$$L_o(x, \vec{\omega}') = L_e(x, \vec{\omega}') + \int_{\Omega} L_i(x, \vec{\omega}) f_r(x, \vec{\omega}, \vec{\omega}') (\vec{\omega} \cdot \vec{n}) d\vec{\omega}$$

- This is an *integral equation*
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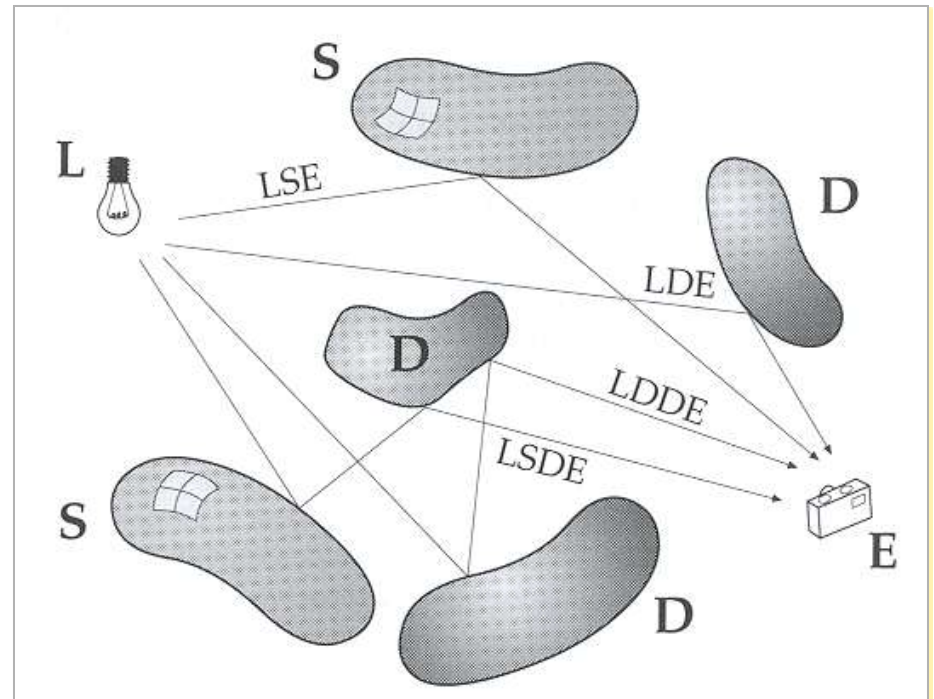
Monte Carlo Integration



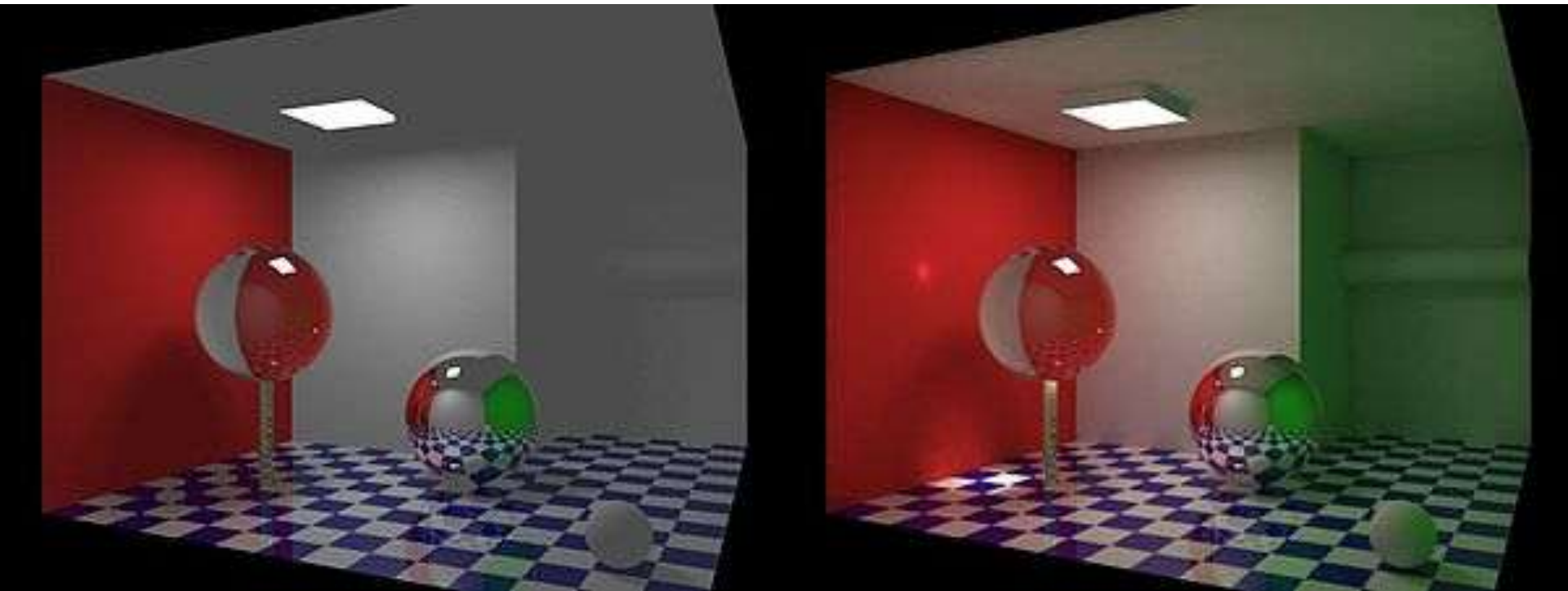
$$\int_0^1 f(x) dx \approx \frac{1}{N} \sum_{i=1}^N f(x_i)$$

Monte Carlo Path Tracing

Estimate integral
for each pixel
by random sampling



Global Illumination



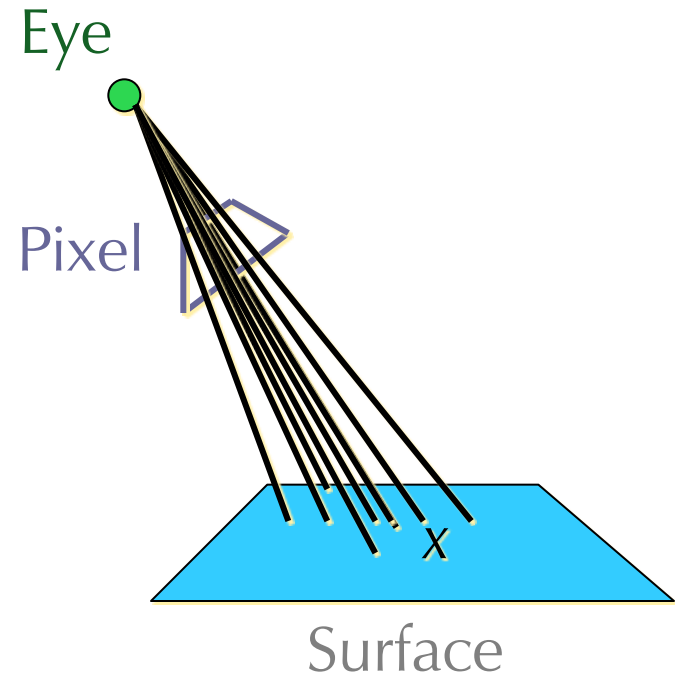
From Grzegorz Tanski, Wikipedia

Monte Carlo Global Illumination

- Rendering = integration
 - Antialiasing
 - Soft shadows
 - Indirect illumination
 - Caustics

Monte Carlo Global Illumination

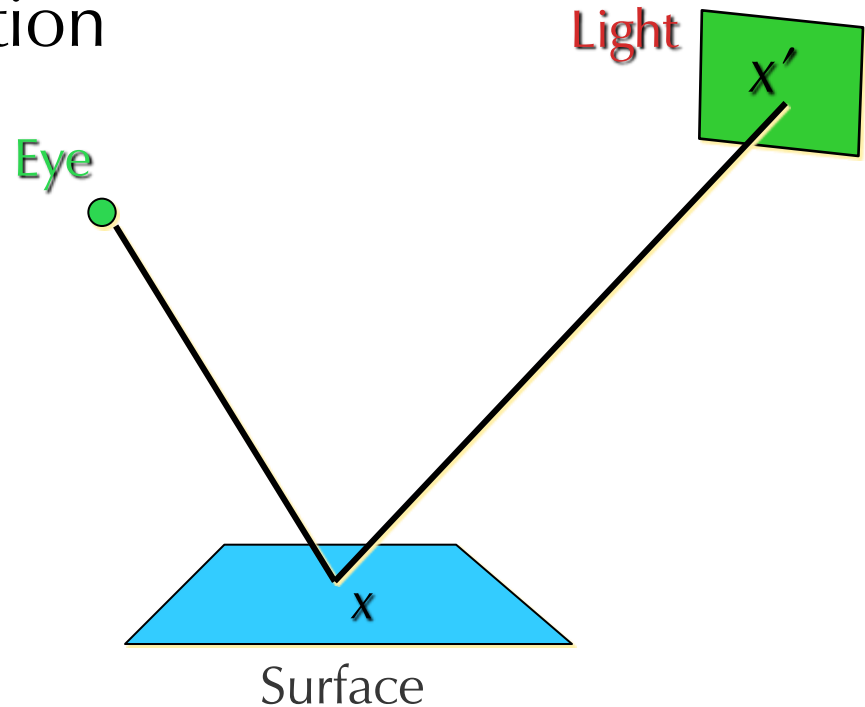
- Rendering = integration
 - Antialiasing
 - Soft shadows
 - Indirect illumination
 - Caustics



$$L_p = \int_S L(x \rightarrow e) dA$$

Monte Carlo Global Illumination

- Rendering = integration
 - Antialiasing
 - Soft shadows
 - Indirect illumination
 - Caustics



$$L(x, \vec{w}) = L_e(x, x \rightarrow e) + \int_S f_r(x, x' \rightarrow x, x \rightarrow e) L(x' \rightarrow x) V(x, x') G(x, x') dA$$

Monte Carlo Global Illumination

- Rendering = integration
 - Antialiasing
 - Soft shadows
 - Indirect illumination
 - Caustics

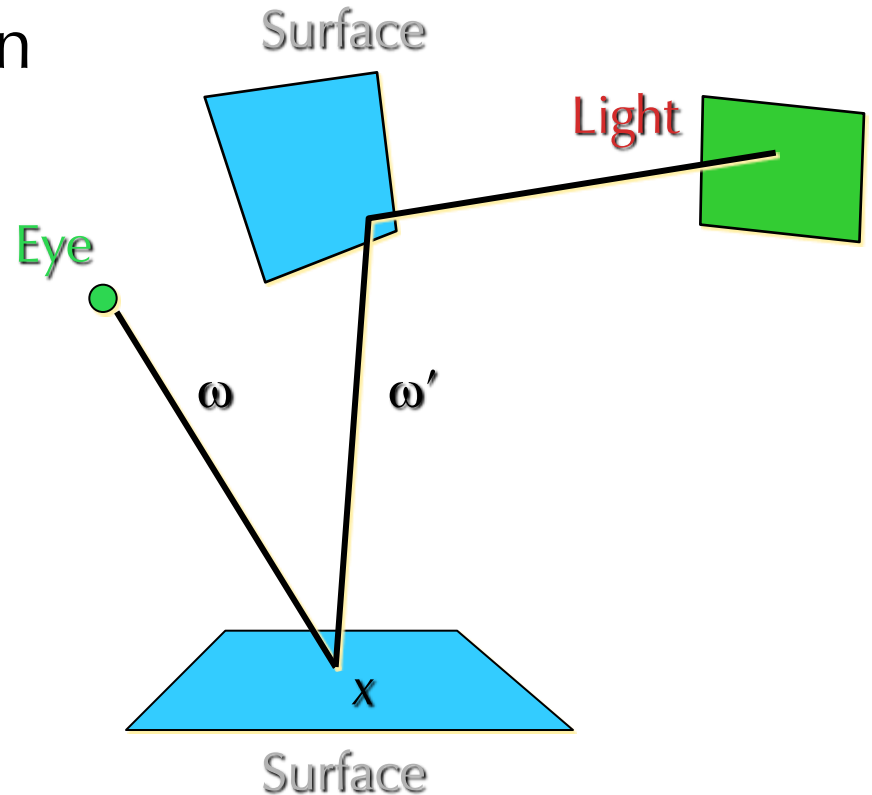


Herf

$$L(x, \vec{w}) = L_e(x, x \rightarrow e) + \int_S f_r(x, x' \rightarrow x, x \rightarrow e) L(x' \rightarrow x) V(x, x') G(x, x') dA$$

Monte Carlo Global Illumination

- Rendering = integration
 - Antialiasing
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$$L_o(x, \vec{\omega}) = L_e(x, \vec{\omega}) + \int_{\Omega} f_r(x, \vec{\omega}', \vec{\omega}) L_i(x, \vec{\omega}') (\vec{\omega}' \cdot \vec{n}) d\vec{\omega}$$

Monte Carlo Global Illumination

- Rendering = integration
 - Antialiasing
 - Soft shadows
 - Indirect illumination
 - Caustics

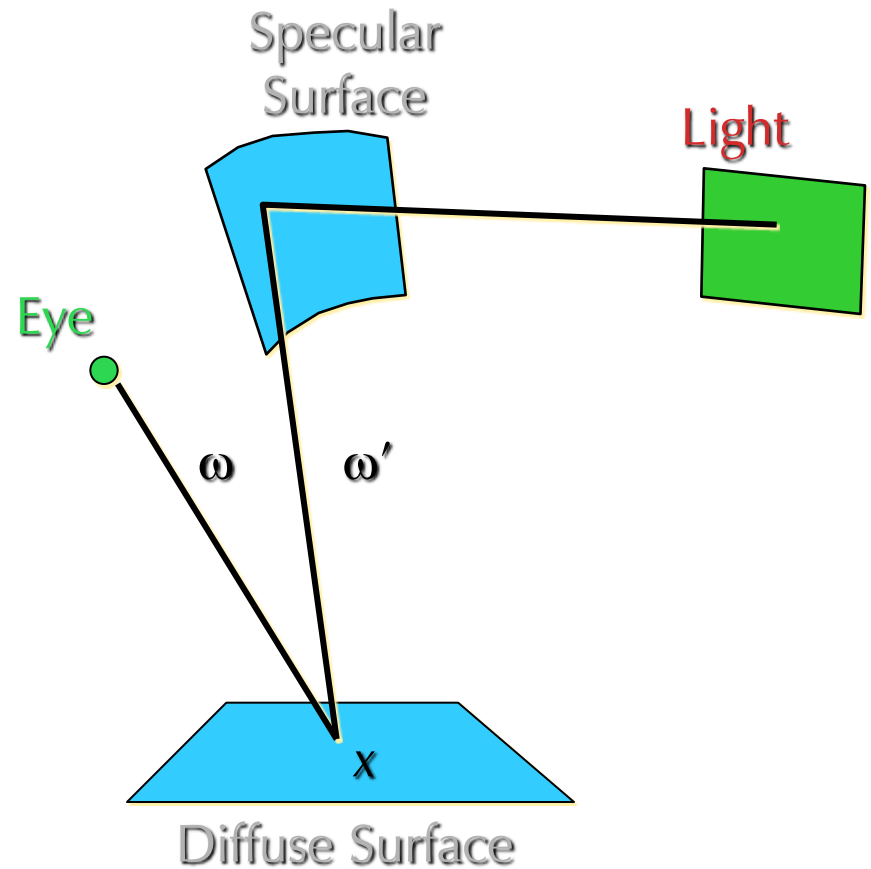


Debevec

$$L_o(x, \vec{w}) = L_e(x, \vec{w}) + \int_{\Omega} f_r(x, \vec{w}', \vec{w}) L_i(x, \vec{w}') (\vec{w}' \cdot \vec{n}) d\vec{w}$$

Monte Carlo Global Illumination

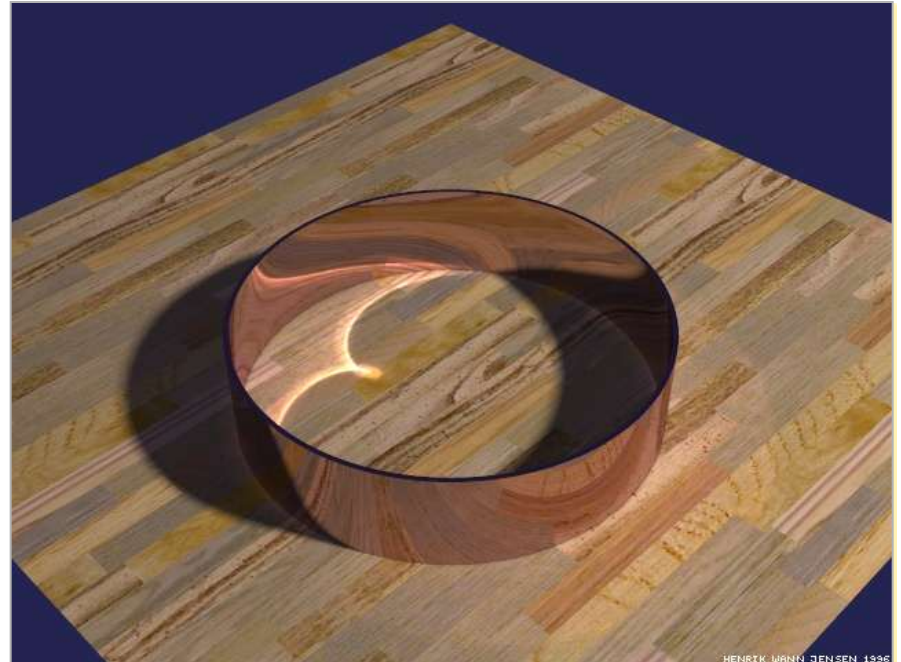
- Rendering = integration
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$$L_o(x, \vec{w}) = L_e(x, \vec{w}) + \int_{\Omega} f_r(x, \vec{w}', \vec{w}) L_i(x, \vec{w}') (\vec{w}' \cdot \vec{n}) d\vec{w}$$

Monte Carlo Global Illumination

- Rendering = integration
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 - Soft shadows
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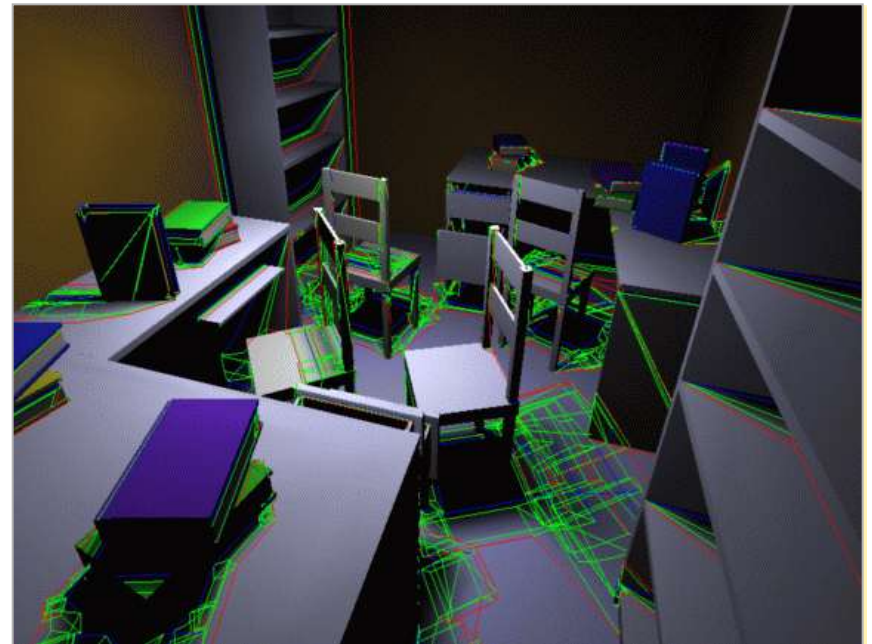
HENRIK WANN JENSEN 1995

Jensen

$$L_o(x, \vec{w}) = L_e(x, \vec{w}) + \int_{\Omega} f_r(x, \vec{w}', \vec{w}) L_i(x, \vec{w}') (\vec{w}' \cdot \vec{n}) d\vec{w}$$

Challenge

- Rendering integrals are difficult to evaluate
 - Multiple dimensions
 - Discontinuities
 - Partial occluders
 - Highlights
 - Caustics



Drettakis

$$L(x, \vec{w}) = L_e(x, x \rightarrow e) + \int_s f_r(x, x' \rightarrow x, x \rightarrow e) L(x' \rightarrow x) V(x, x') G(x, x') dA$$

Challenge

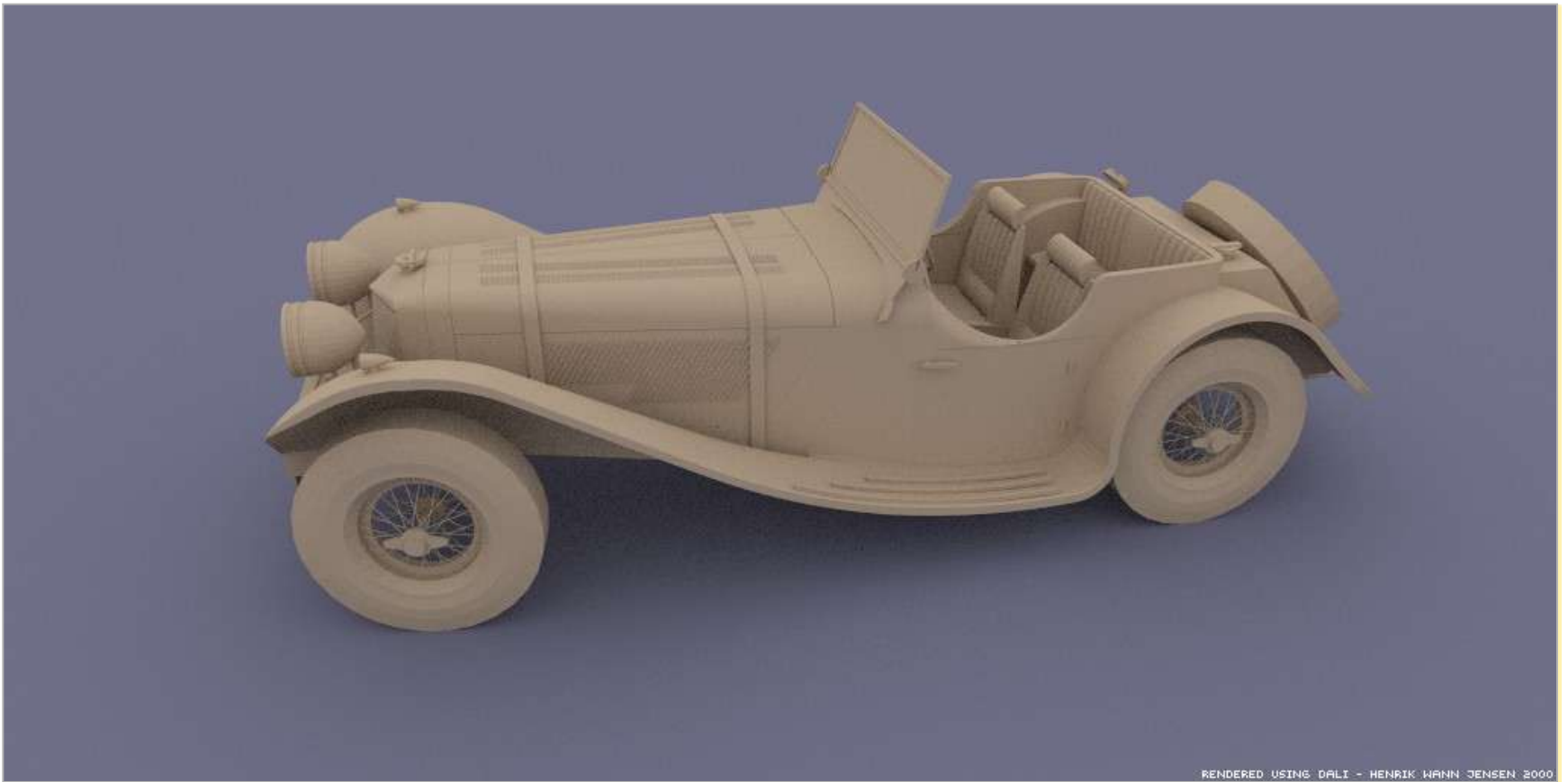
- Rendering integrals are difficult to evaluate
 - Multiple dimensions
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Jensen

$$L(x, \vec{w}) = L_e(x, x \rightarrow e) + \int_S f_r(x, x' \rightarrow x, x \rightarrow e) L(x' \rightarrow x) V(x, x') G(x, x') dA$$

Monte Carlo Path Tracing



Big diffuse light source, 20 minutes

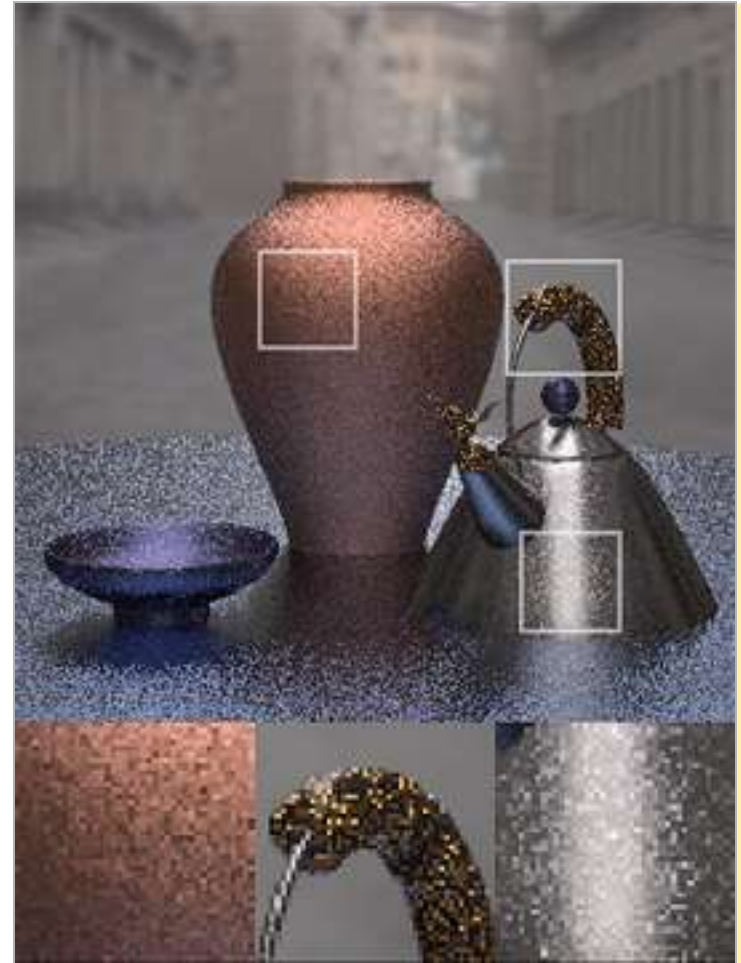
Monte Carlo Path Tracing



1000 paths/pixel

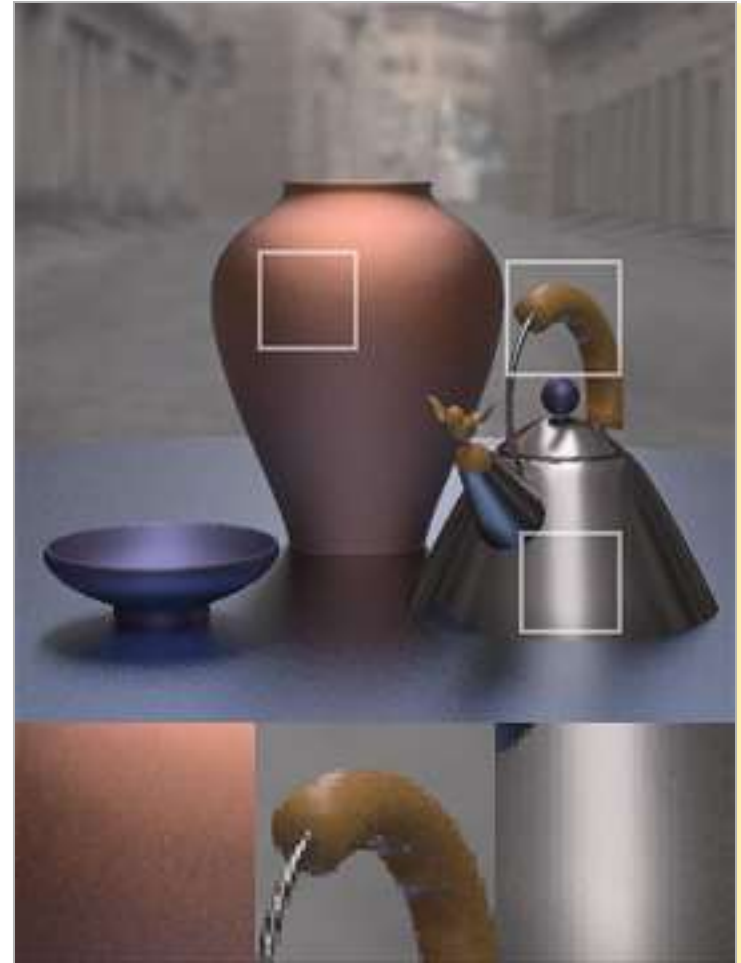
Monte Carlo Path Tracing

- Drawback: can be noisy unless *lots* of paths simulated
- 40 paths per pixel:



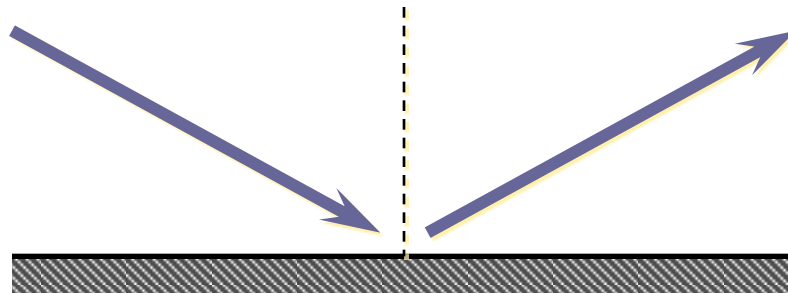
Monte Carlo Path Tracing

- Drawback: can be noisy unless *lots* of paths simulated
- 1200 paths per pixel:



Reducing Variance

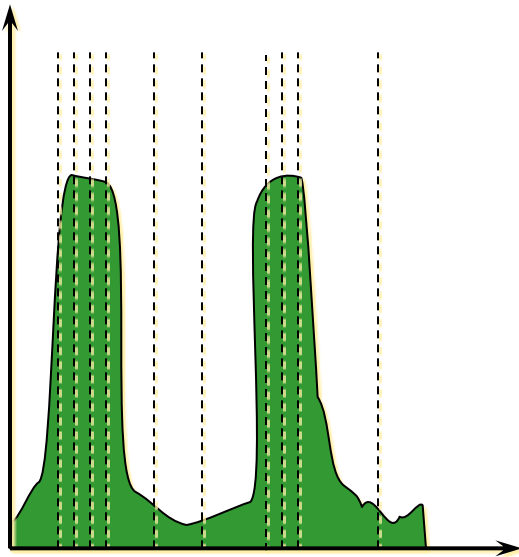
- Observation: some paths more important (carry more energy) than others
 - For example, shiny surfaces reflect more light in the ideal “mirror” direction



- Idea: put more samples where $f(x)$ is bigger

Importance Sampling

- Idea: put more samples where $f(x)$ is bigger

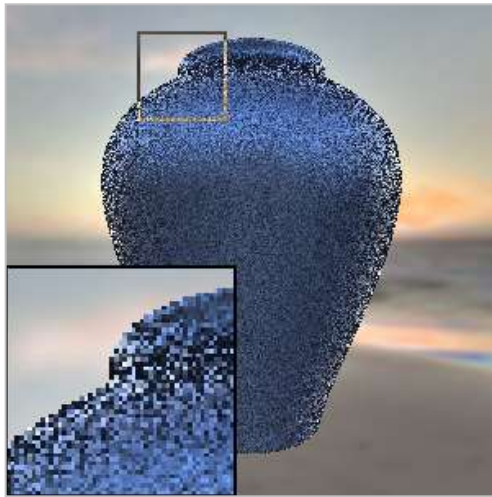


$$\int_0^1 f(x) dx = \frac{1}{N} \sum_{i=1}^N Y_i$$

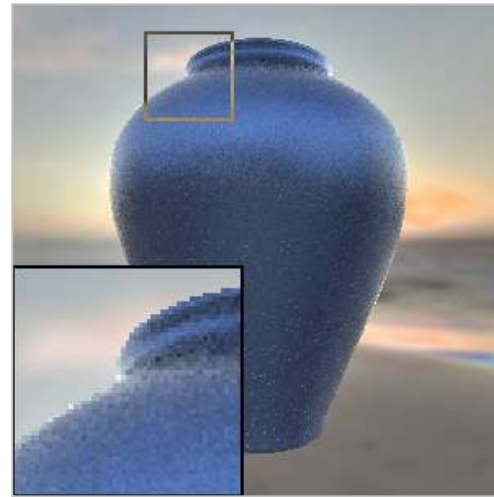
$$Y_i = \frac{f(x_i)}{p(x_i)}$$

Effect of Importance Sampling

- Less noise at a given number of samples



Uniform random sampling



Importance sampling

- Equivalently, need to simulate fewer paths for some desired limit of noise

Other examples

- <http://www.jposhea.org/projects/monte/>

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- More information on Monte Carlo:

<http://arxiv.org/abs/hep-ph/0006269/>

- A review of Monte Carlo Ray Tracing Methods:

[http://www.cg.tuwien.ac.at/~balazs/PAPERS/CESC
G97/mcrt.html](http://www.cg.tuwien.ac.at/~balazs/PAPERS/CESC
G97/mcrt.html)

Matlab functions

- rand, randi, randn (normal)
- rng: Configure your random number generator!
- Quasi-random: haltonset, sobolset