Constrained Optimization

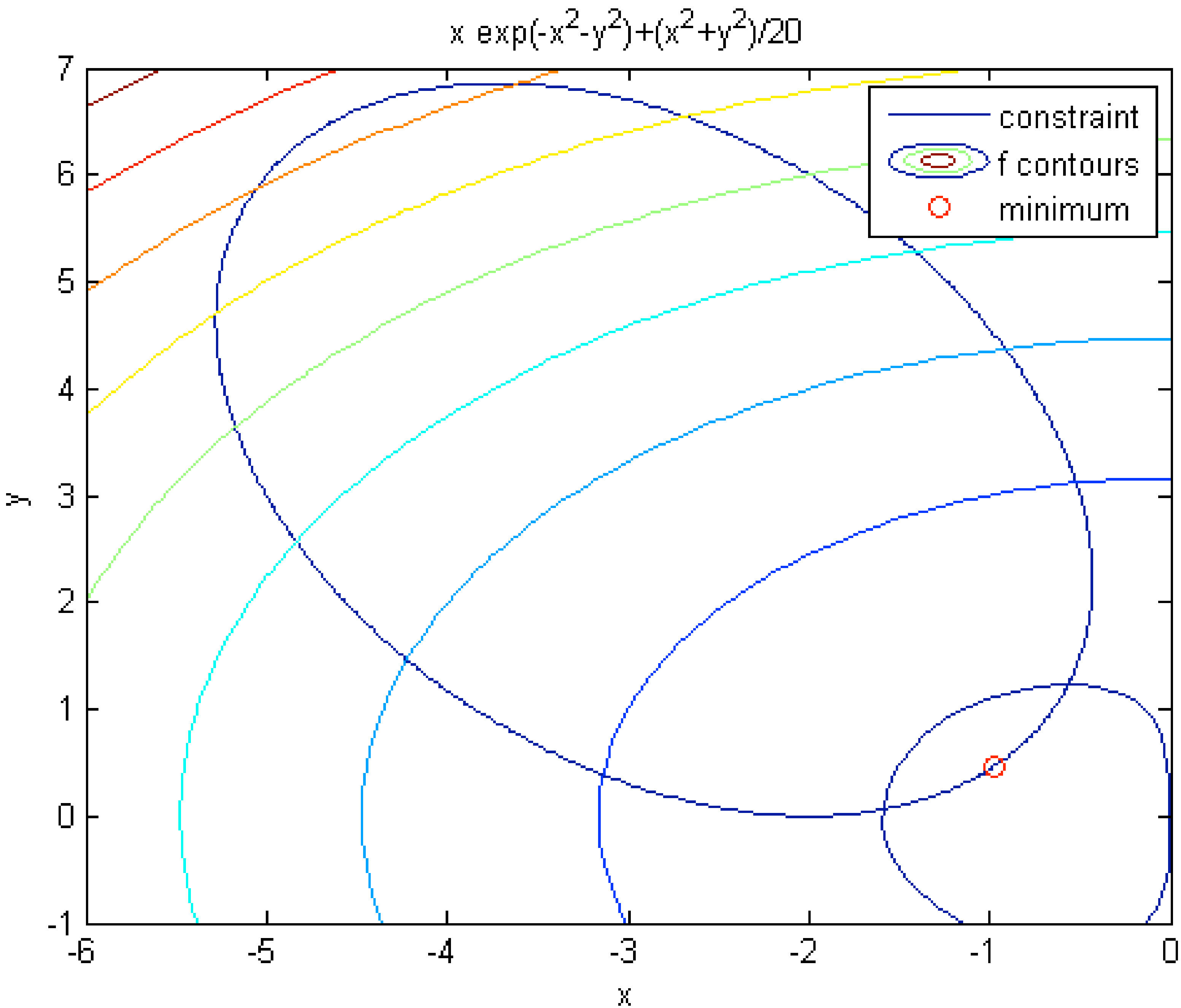
COS 323

Last time

- Introduction to optimization
  - objective function, variables, [constraints]
- 1-dimensional methods
  - Golden section, discussion of error
  - Newton's method
- Multi-dimensional methods
  - Newton's method, steepest descent, conjugate gradient
- General strategies, value-only methods

Today

- Linear constrained optimization
  - Linear programming (LP)
  - Simplex method for LP
- General optimization
  - With equality constraints: Lagrange multipliers
  - With inequality: KKT conditions + Quadratic programming


x exp(-x^2-y^2)+(x^2+y^2)/20
constraint
f contours
minimum
y
x
7
6
5
4
3
2
1
0
-1
-6
-5
-4
-3
-2
-1
0

Linear Programming: Linear Objective + Linear Constraints

Standard form: maximize objective

$$\zeta = c_1 x_1 + c_2 x_2 + \cdots$$

with ***primary*** constraints

$$x_1 \geq 0,\ x_2 \geq 0,\ \cdots$$

and additional constraints

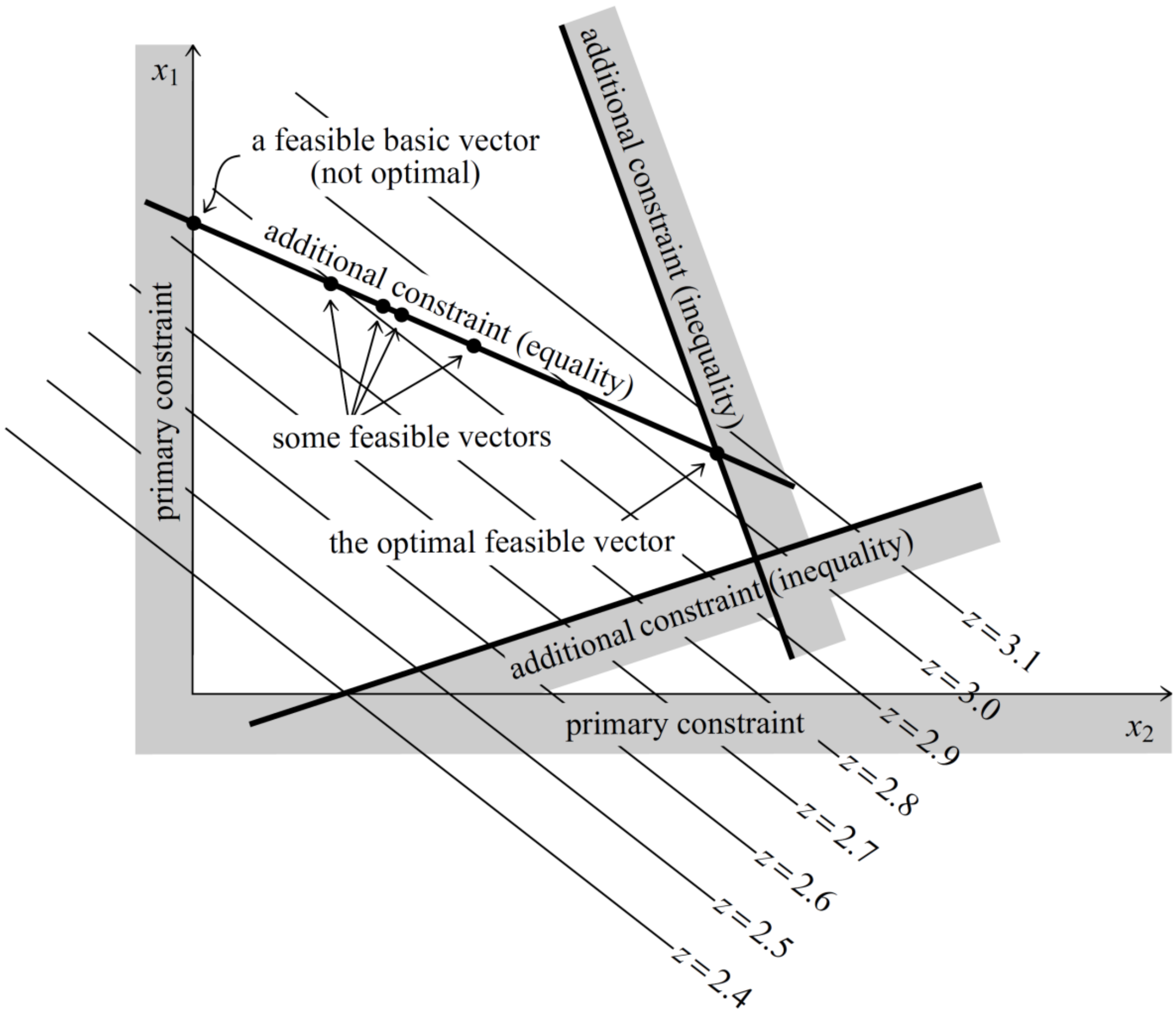
$$a_{11} x_1 + a_{12}\, x_2 + \cdots \leq b_1$$
$$a_{21} x_1 + a_{22}\, x_2 + \cdots \leq b_2$$

Why Linear Programming?

- What is the cheapest combination of foods that meets your nutritional needs?
- How do you minimize the risk in your investment portfolio, subject to achieving a given return?
- How should an airline assign crews to flights?
- Traveling salesman problem

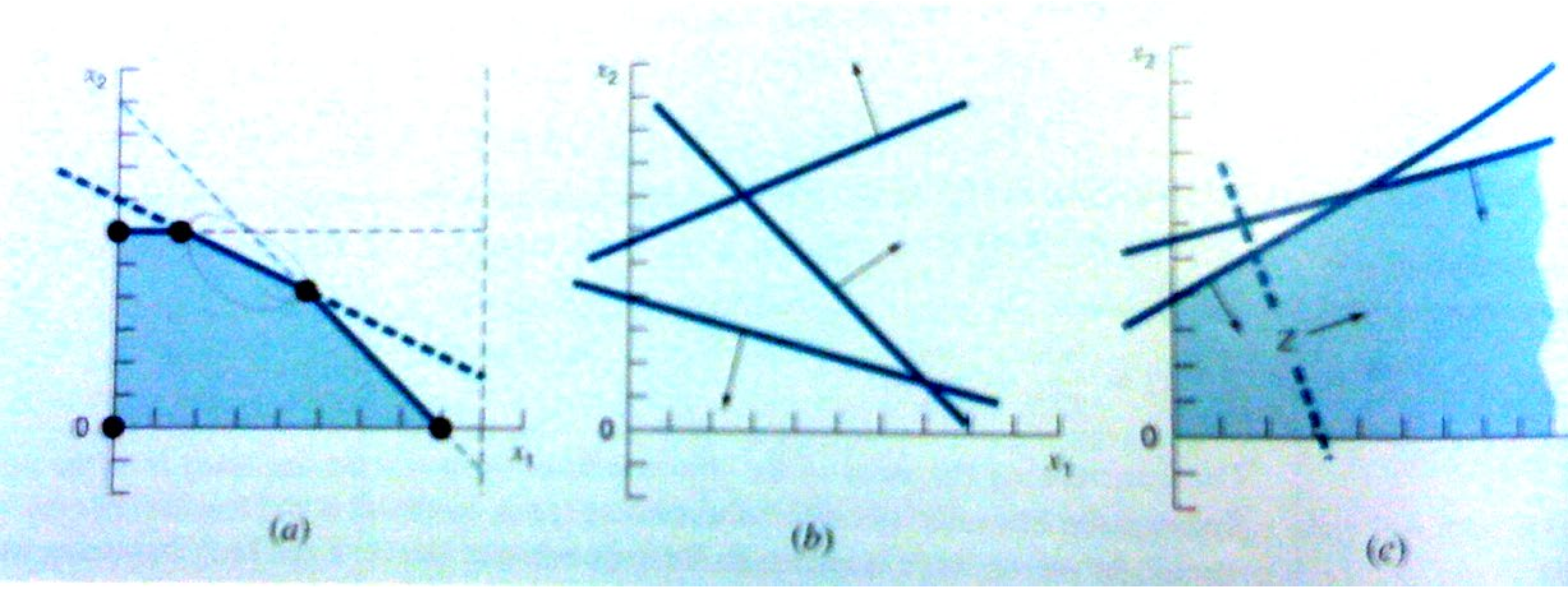
from John Mitchell @ RPI

Linear Objective + Linear Constraints



[Press et al.]

Other possible outcomes



Simplex Method

- [Dantzig]'s Simplex Method
- Basic idea:
  - Phase I: Find a "basic feasible solution":
    A vertex satisfying all constraints.
  - Phase II: Traverse vertices of the polytope along edges for which objective function is *strictly decreasing*
    - At any vertex, some *n* constraints are satisfied with exact equality

Simplex Form

- Transform each inequality constraint to an equality constraint using *slack variables*

$$x_1 + 2x_3 \leq 740 \qquad\qquad x_1 + 2x_3 + y_1 = 740$$

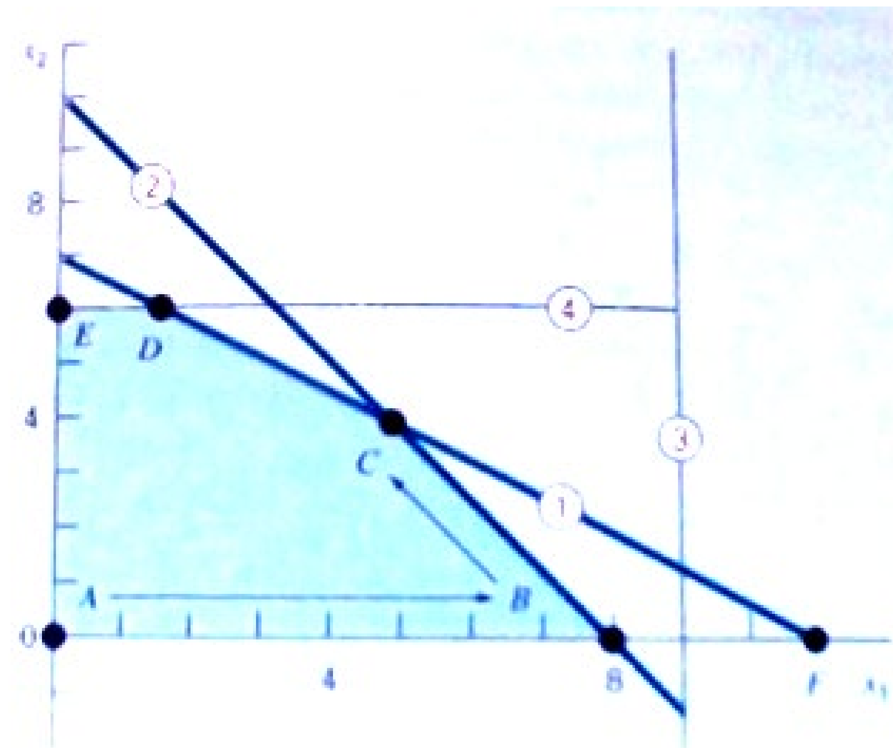
$$2x_2 - 7x_4 \leq 0 \qquad\qquad 2x_2 - 7x_4 + y_2 = 0$$

$$x_2 - x_3 + 2x_4 \geq \frac{1}{2} \qquad\qquad x_2 - x_3 + 2x_4 - y_3 = \frac{1}{2}$$

$$x_1 + x_2 + x_3 + x_4 = 9 \qquad\qquad x_1 + x_2 + x_3 + x_4 = 9$$

An underdetermined system

- Normalized form:
  m equations with *n* unknowns,
  with m < n
- Solve by setting (n – m) variables to 0 and solving for remaining m unknowns

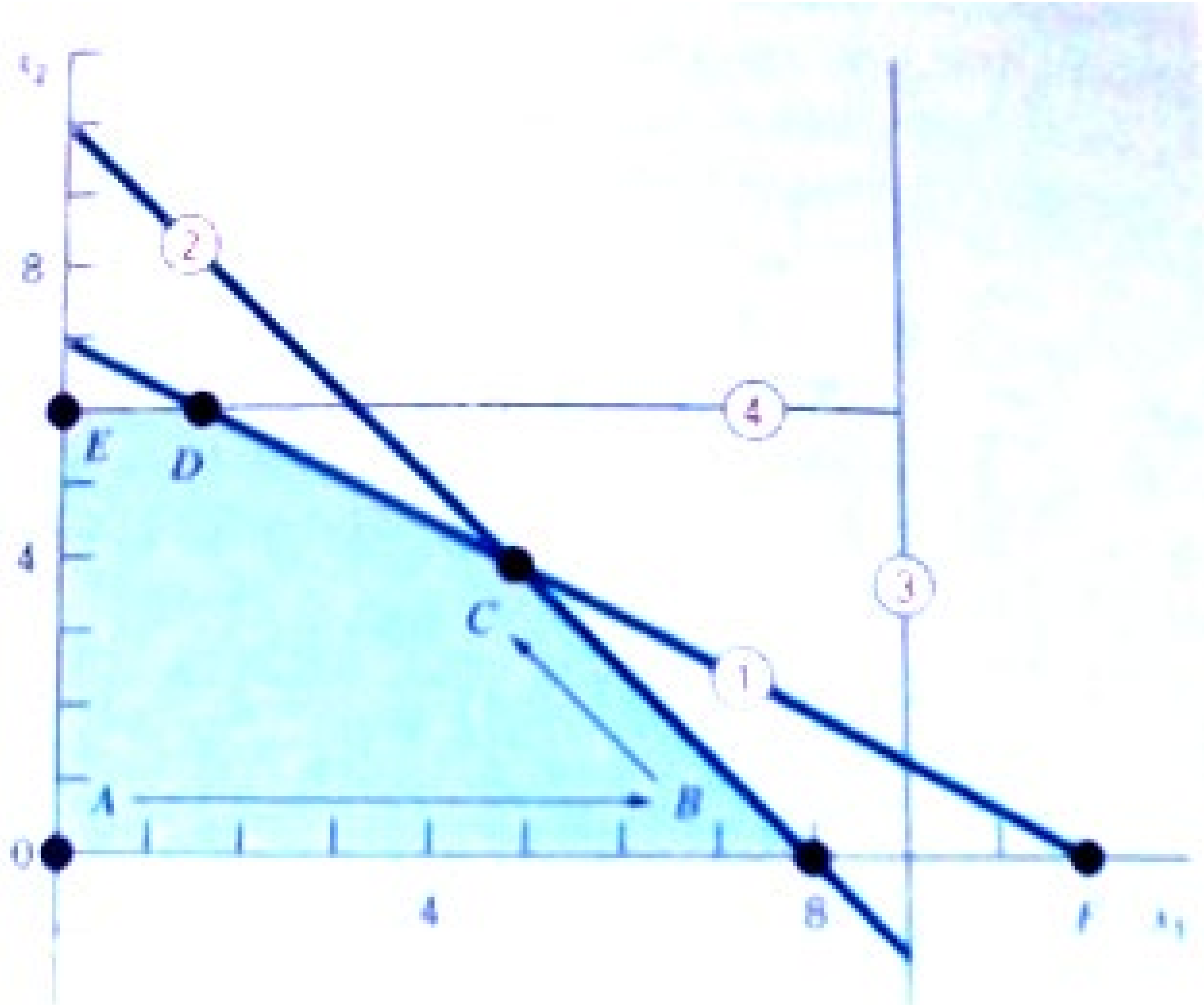


$$C_m^n = \frac{n!}{m!(n-m)!}$$ systems of equations ☹

Simplex method vertex traversal

- Start with a *basic feasible solution*
  - Set (n-m) variables to 0: "non-basic variables"
  - Solve for remaining *m* "basic variables":
    If they're ≥ 0, then it's feasible
- Traverse an edge:
  - "Swap" a basic w/ a non-basic variable
  - Increase the value of the basic variable that has the biggest potential impact on the objective, and increase until another constraint is encountered (i.e., leaving basic variable becomes 0)

Results of simplex method



For more information

- *Numerical Recipes in C* chapter on linear programming: *http://www.nrbook.com/a/bookcpdf/c10-8.pdf*
- Interactive Java applet: http://campuscgi.princeton.edu/~rvdb/JAVA/pivot/simple.html

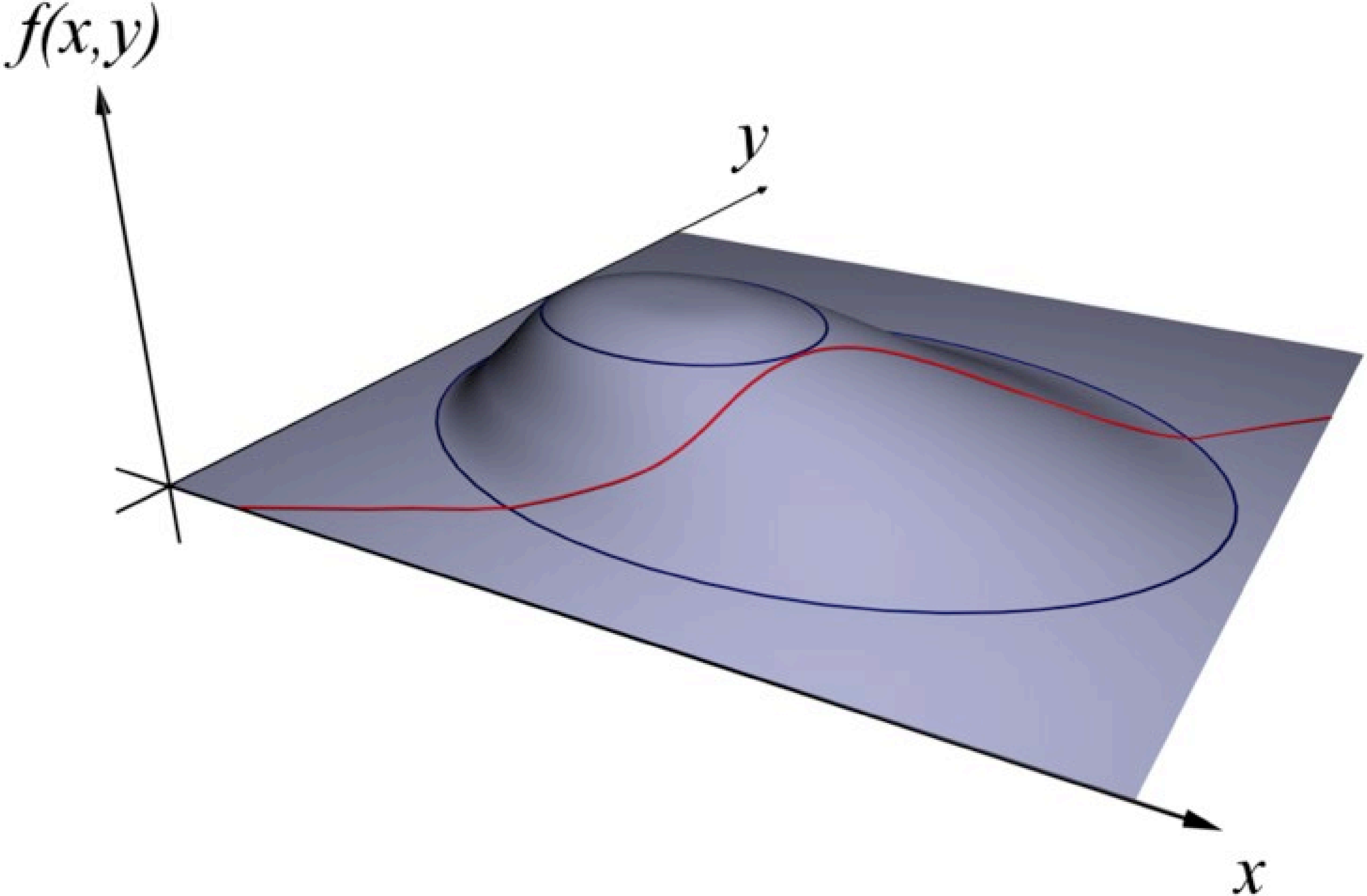
Comments on Simplex Method

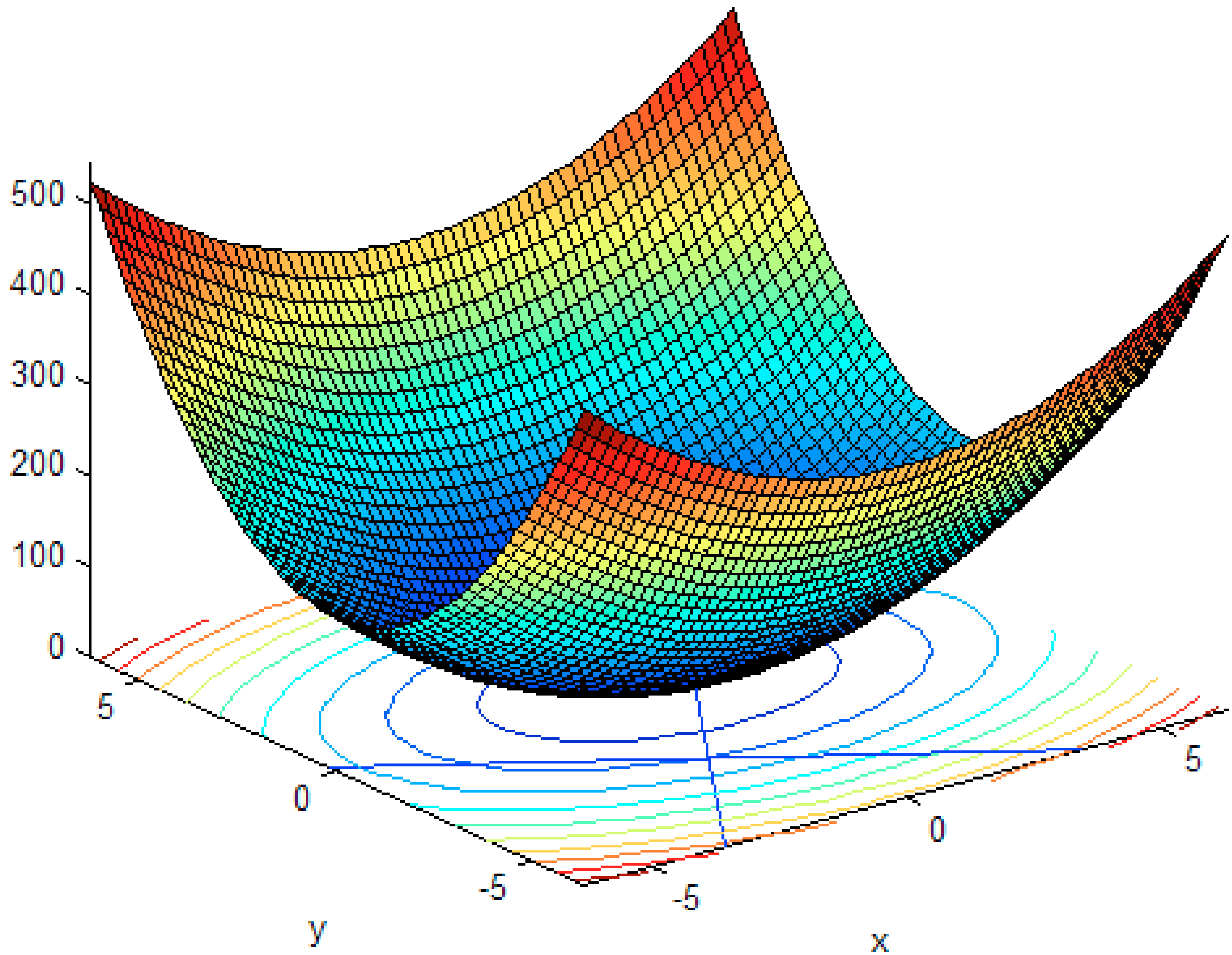
- In theory: can take very long – exponential in the input length
- In practice: efficient – # of iterations typically a few times # of constraints
  - Should take care to detect cycles
- There exist provably polynomial-time algorithms

Beyond linear optimization

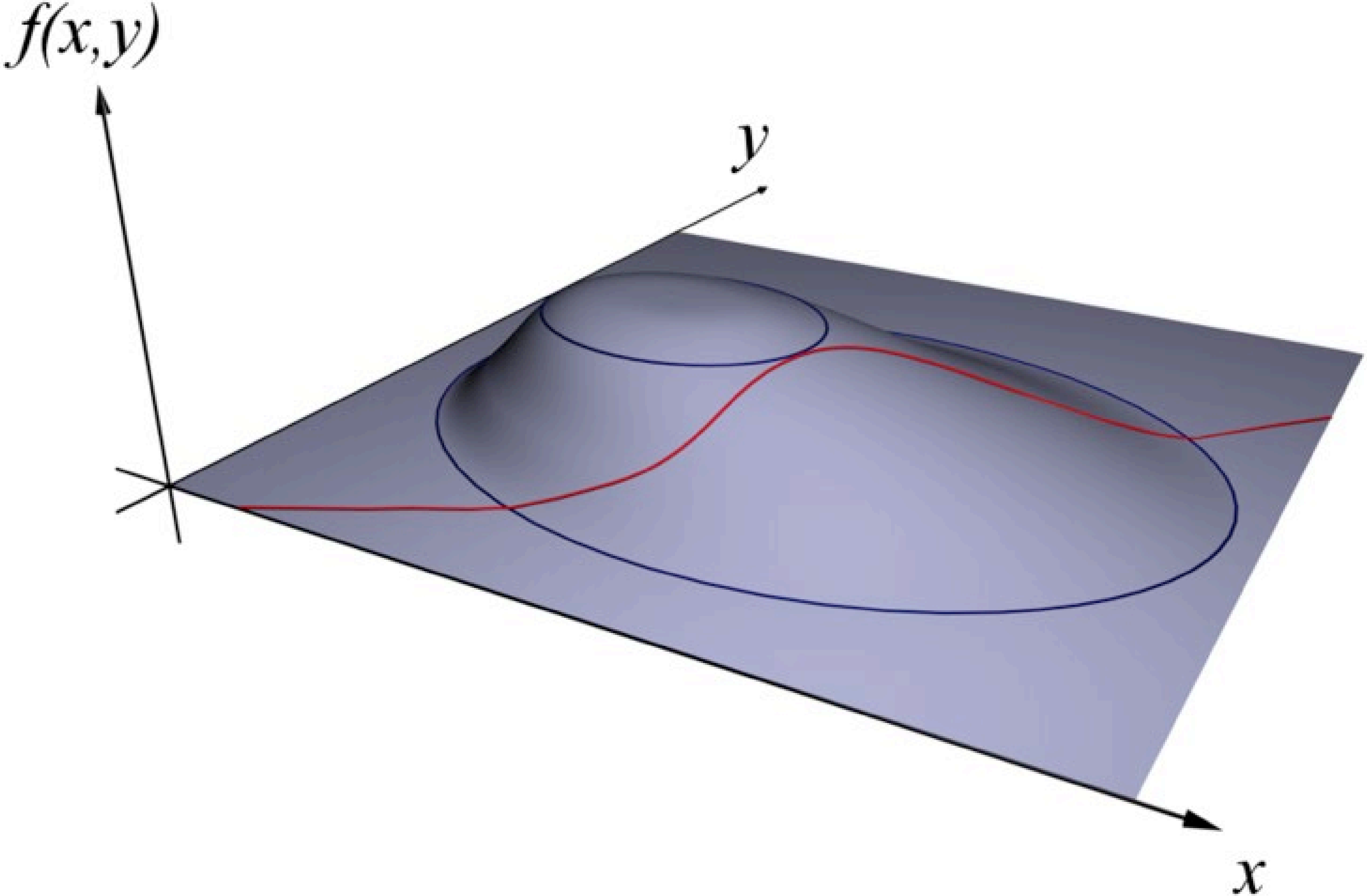
General Optimization with Equality Constraints

- Minimize $f(x)$ subject to $g_i(x) = 0$

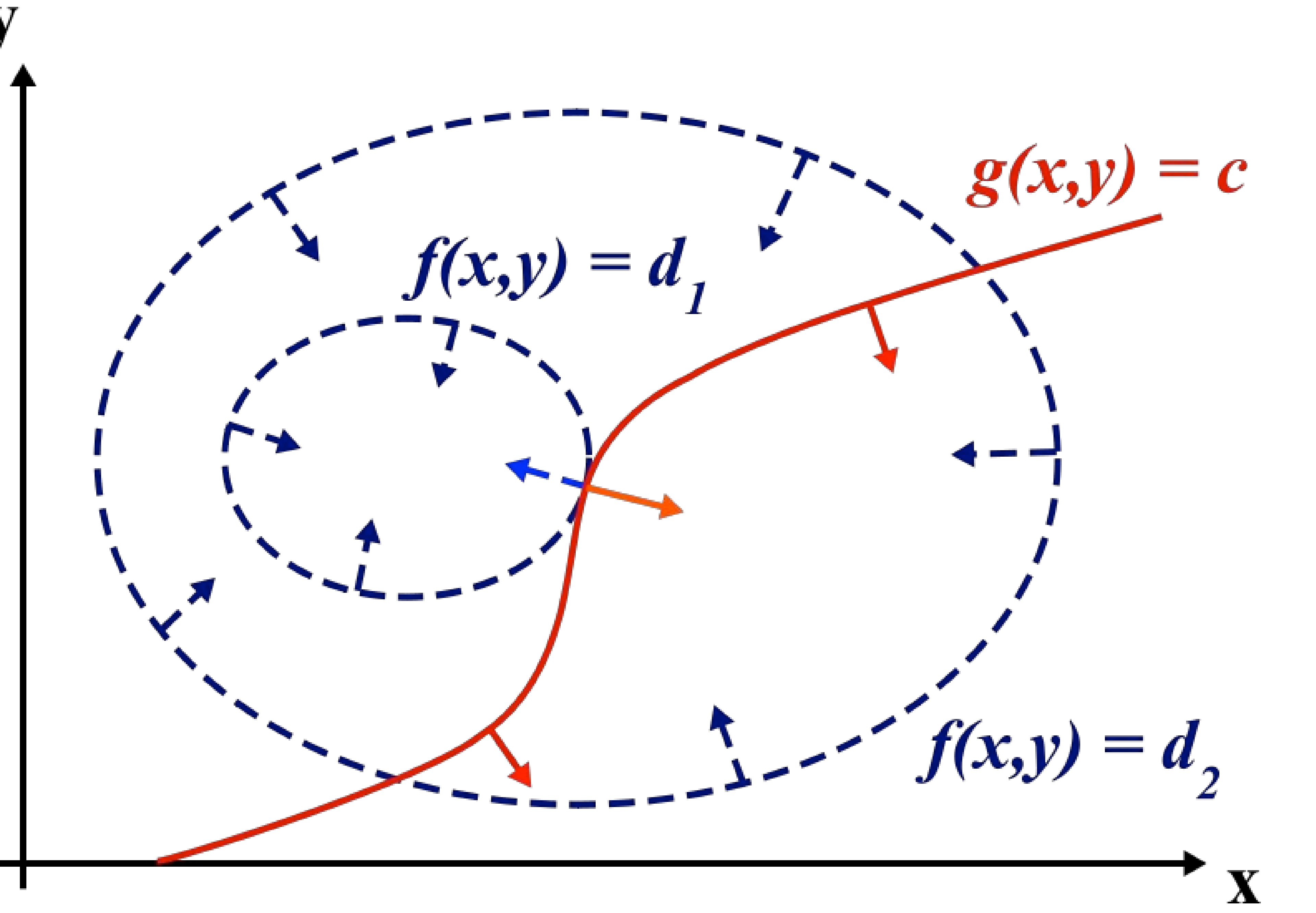

f(x,y)
y
x



500
400
300
200
100
0
5
0
-5
y
-5
0
5
x



f(x,y)
y
x

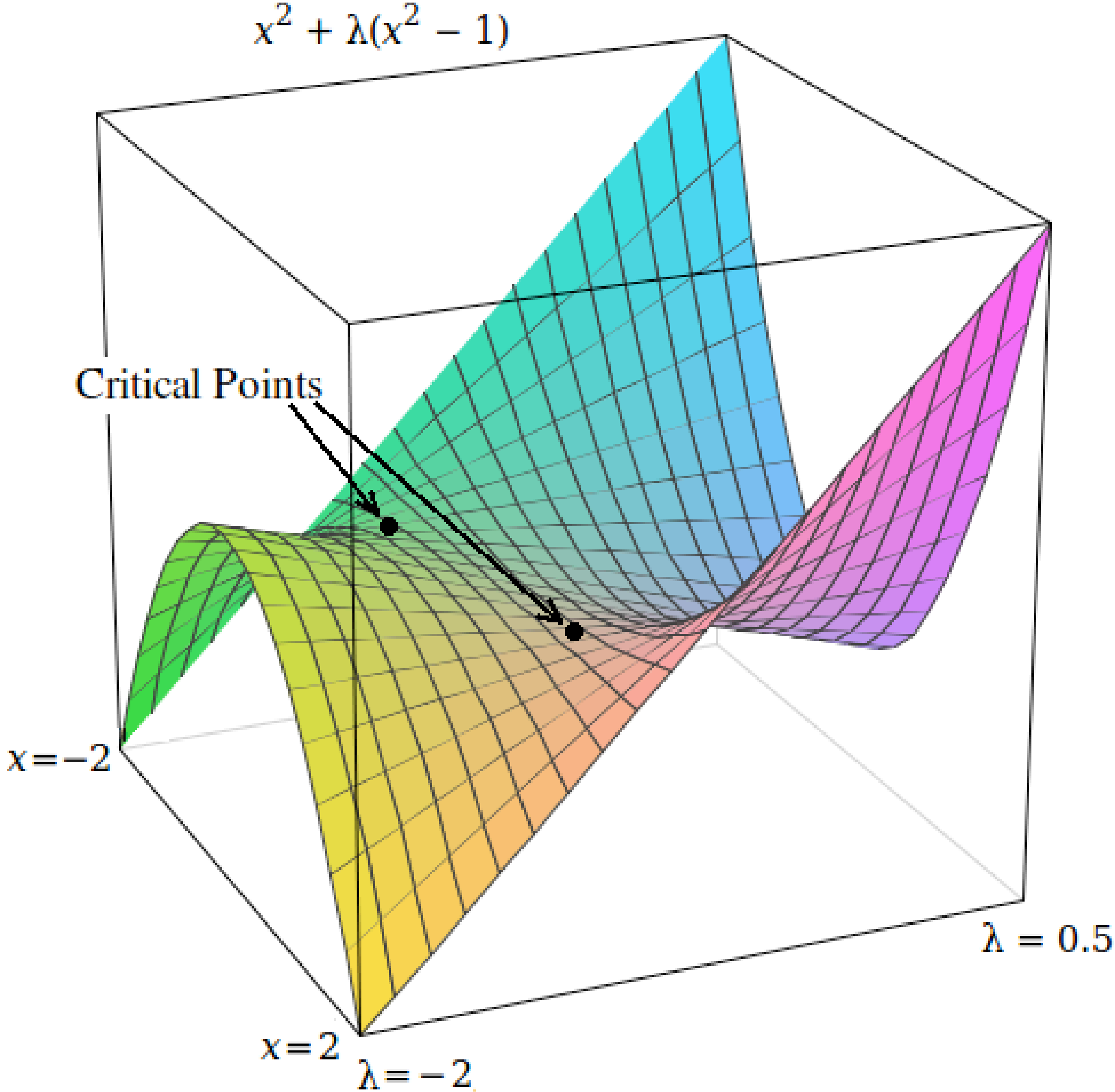


y
g(x,y) = c
f(x,y) = d1
f(x,y) = d2
x

General Optimization with Equality Constraints

- Minimize $f(x)$ subject to $g_i(x) = 0$
- Method of Lagrange multipliers: convert to a higher-dimensional problem
- Find **critical points** of $f(x) + \sum \lambda_i g_i(x)$

  w.r.t. $(x_1 \ldots x_n; \lambda_1 \ldots \lambda_k)$



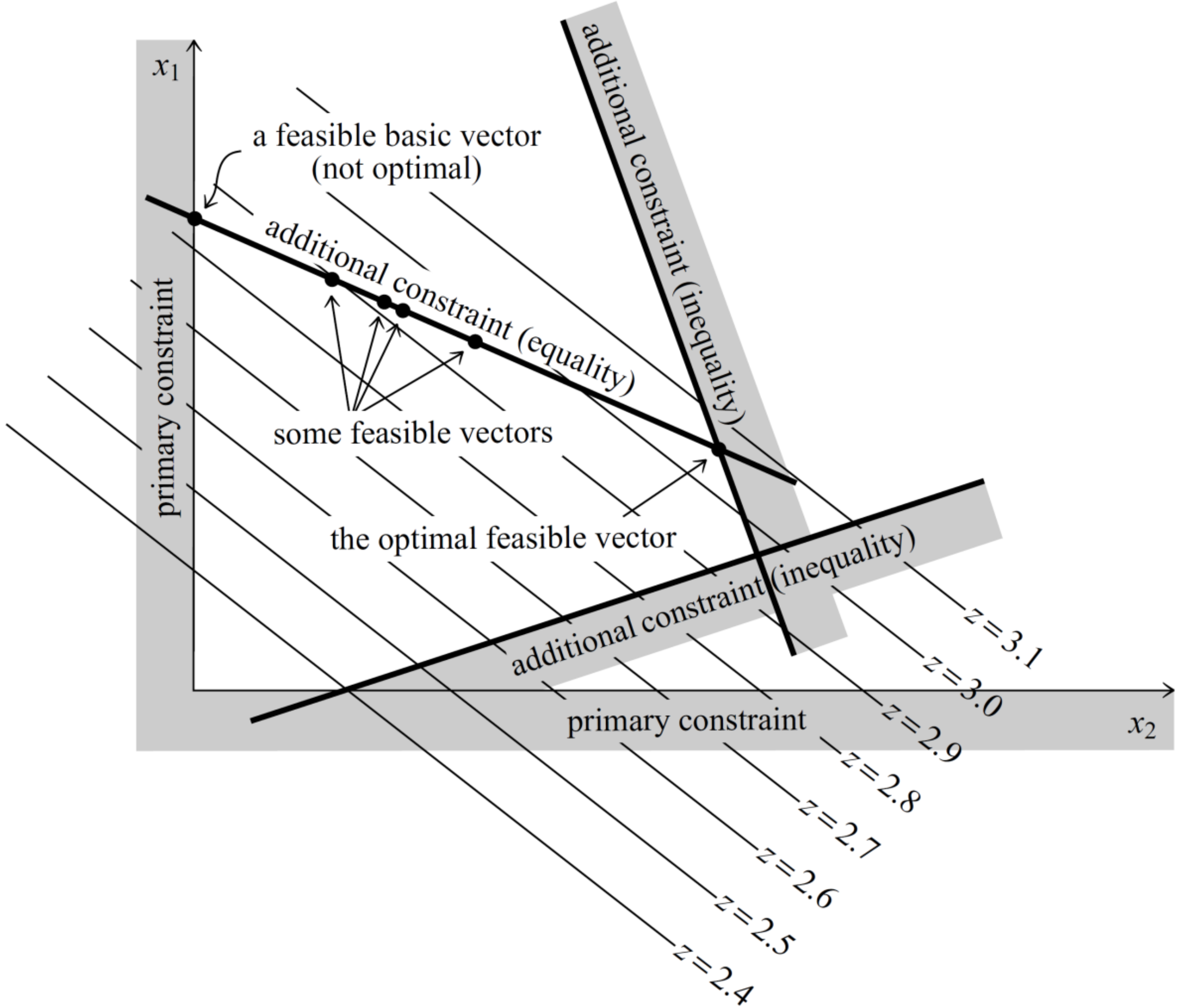
$x^2 + \lambda(x^2 - 1)$
Critical Points
$x = -2$
$x = 2$
$\lambda = -2$
$\lambda = 0.5$

Comments on Lagrange Multipliers

- A way of re-defining an optimization problem in terms of a *necessary condition* for optimality
  - *not* an algorithm for finding optimal points!
- Use other method to find critical points
- Sometimes lambdas are interesting in themselves
  - Lagrangian mechanics
  - “Shadow pricing” in economics: The “marginal cost” of a constraint

General Optimization with Inequality Constraints

- Minimize $f(x)$ subject to $h_i(x) \leq 0$
- More complicated: some inequality constraints might be irrelevant

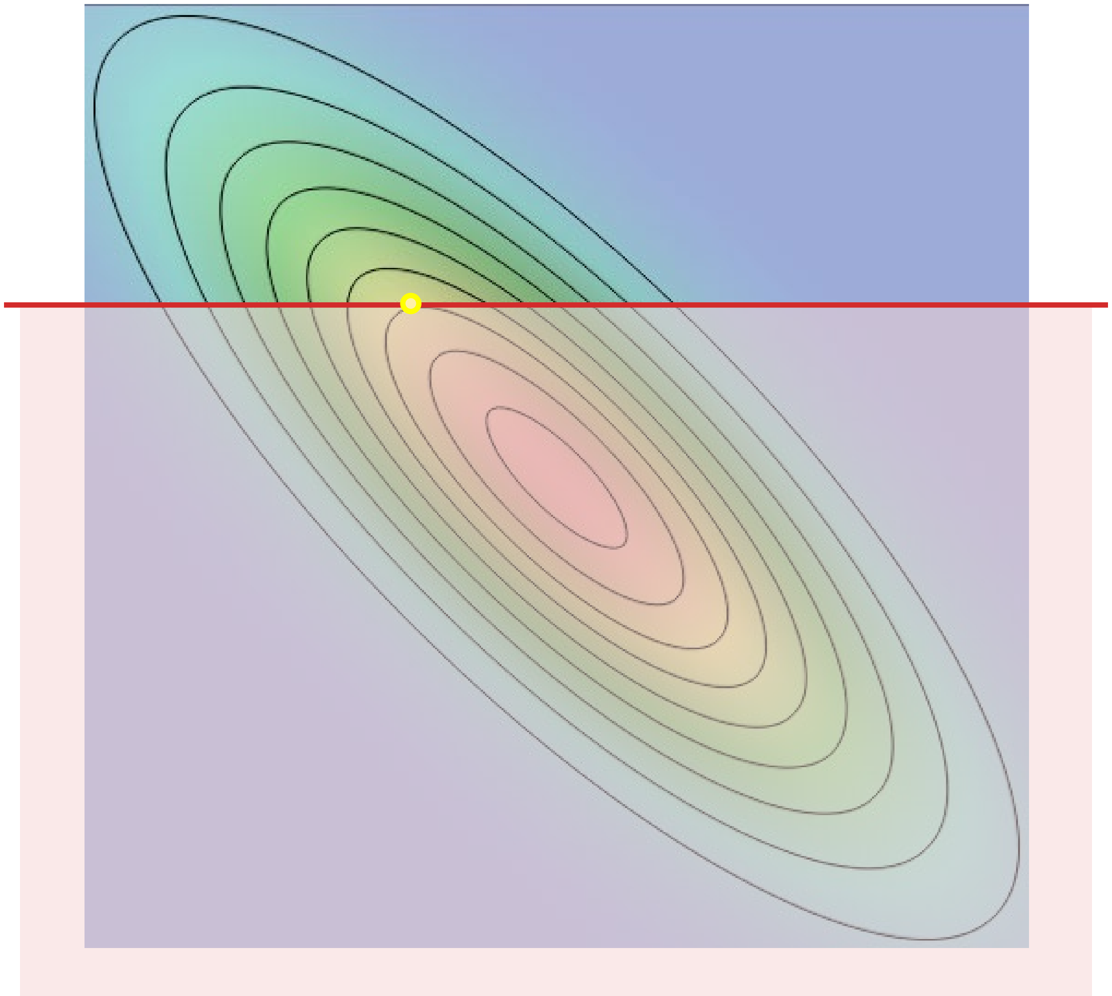

x_1
x_2
a feasible basic vector
(not optimal)
additional constraint (equality)
additional constraint (inequality)
additional constraint (inequality)
primary constraint
primary constraint
some feasible vectors
the optimal feasible vector
$z=3.1$
$z=3.0$
$z=2.9$
$z=2.8$
$z=2.7$
$z=2.6$
$z=2.5$
$z=2.4$

KKT Conditions for optimization with inequality constraints

- Karush-Kuhn-Tucker (KKT) conditions
  - Generalization of Lagrange multipliers
- Minimize $f(x)+\sum \lambda_i h_i(x)$ w.r.t. $(x_1 \ldots x_n; \lambda_1 \ldots \lambda_k)$
- Subject to

$$h_i(x) \leq 0, \quad \lambda_i(x) \geq 0, \quad \lambda_i(x)\, h_i(x) = 0$$

KKT Conditions



minimize f(x)

with h(x) ≤ 0

1. $\partial / \partial x \,(f(x) + \lambda h(x)) = 0$
2. $\partial / \partial \lambda \,(f(x) + \lambda h(x)) = 0$
3. $h(x) \leq 0$
4. $\lambda \geq 0$
5. $\lambda\, h(x) = 0$

Quadratic Programming

- The KKT conditions allow writing a system with quadratic objective and linear constraints as a linear program
  - Solve with simplex, etc.

In practice

- Matlab:
  - **linprog**
    - Option of using simplex method
  - **quadprog**
    - Option of getting Lagrange multipler values out
  - See Matlab's official advice on choosing a solver
    - http://www.mathworks.com/help/toolbox/optim/ug/brhkghv-18.html#bsbqd7i
- Excel solver