5.2 Tries

- R-way tries
- ternary search tries
- character-based operations
Summary of the performance of symbol-table implementations

Order of growth of the frequency of operations.

<table>
<thead>
<tr>
<th>implementation</th>
<th>typical case</th>
<th>ordered operations</th>
<th>operations on keys</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>search</td>
<td>insert</td>
<td>delete</td>
</tr>
<tr>
<td>red-black BST</td>
<td>log N</td>
<td>log N</td>
<td>log N</td>
</tr>
<tr>
<td>hash table</td>
<td>1†</td>
<td>1†</td>
<td>1†</td>
</tr>
</tbody>
</table>

† under uniform hashing assumption

Q. Can we do better?
A. Yes, if we can avoid examining the entire key, as with string sorting.
String symbol table basic API

**String symbol table.** Symbol table specialized to string keys.

```java
public class StringST<Value> {
    StringST(); // create an empty symbol table
    void put(String key, Value val); // put key-value pair into the symbol table
    Value get(String key); // return value paired with given key
    void delete(String key); // delete key and corresponding value
}
```

**Goal.** Faster than hashing, more flexible than BSTs.
### String symbol table implementations cost summary

<table>
<thead>
<tr>
<th>Implementation</th>
<th>Character accesses (typical case)</th>
<th>Dedup</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td>$c \log^2 N$</td>
</tr>
<tr>
<td>hashing (linear probing)</td>
<td>$L$</td>
<td>$L$</td>
</tr>
</tbody>
</table>

#### Parameters
- $N = \text{number of strings}$
- $L = \text{length of string}$
- $R = \text{radix}$

<table>
<thead>
<tr>
<th>File</th>
<th>Size</th>
<th>Words</th>
<th>Distinct</th>
</tr>
</thead>
<tbody>
<tr>
<td>moby.txt</td>
<td>1.2 MB</td>
<td>210 K</td>
<td>32 K</td>
</tr>
<tr>
<td>actors.txt</td>
<td>82 MB</td>
<td>11.4 M</td>
<td>900 K</td>
</tr>
</tbody>
</table>

**Challenge.** Efficient performance for string keys.
5.2 Tries

- R-way tries
- ternary search tries
- character-based operations
Tries. [from retrieval, but pronounced "try"]

- Store characters in nodes (not keys).
- Each node has $R$ children, one for each possible character.
- For now, we do not draw null links.

<table>
<thead>
<tr>
<th>key</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>by</td>
<td>4</td>
</tr>
<tr>
<td>sea</td>
<td>6</td>
</tr>
<tr>
<td>sells</td>
<td>1</td>
</tr>
<tr>
<td>she</td>
<td>0</td>
</tr>
<tr>
<td>shells</td>
<td>3</td>
</tr>
<tr>
<td>shore</td>
<td>7</td>
</tr>
<tr>
<td>the</td>
<td>5</td>
</tr>
</tbody>
</table>
Search in a trie

Follow links corresponding to each character in the key.

- **Search hit**: node where search ends has a non-null value.
- **Search miss**: reach null link or node where search ends has null value.

```
get("shells")
```
Search in a trie

Follow links corresponding to each character in the key.

- **Search hit:** node where search ends has a non-null value.
- **Search miss:** reach null link or node where search ends has null value.

```
get("she")
```

![Trie diagram](image)
Search in a trie

Follow links corresponding to each character in the key.
- **Search hit:** node where search ends has a non-null value.
- **Search miss:** reach null link or node where search ends has null value.

get("shell")

![Trie diagram](image-url)
Search in a trie

Follow links corresponding to each character in the key.

- **Search hit:** node where search ends has a non-null value.
- **Search miss:** reach null link or node where search ends has null value.

get("shelter")
Insertion into a trie

Follow links corresponding to each character in the key.
- Encounter a null link: create new node.
- Encounter the last character of the key: set value in that node.

```plaintext
put("shore", 7)
```

![Trie diagram](image)
Trie construction demo
Trie representation: Java implementation

**Node.** A value, plus references to $R$ nodes.

```java
private static class Node {
    private Object value;
    private Node[] next = new Node[R];
}
```

characters are implicitly defined by link index

Trie representation

neither keys nor characters are explicitly stored

each node has an array of links and a value

use Object instead of Value since no generic array creation in Java
R-way trie: Java implementation

```java
public class TrieST<Value> {
    private static final int R = 256;
    private Node root = new Node();

    private static class Node {
        /* see previous slide */
    }

    public void put(String key, Value val) {
        root = put(root, key, val, 0);
    }

    private Node put(Node x, String key, Value val, int d) {
        if (x == null) x = new Node();
        if (d == key.length()) { x.val = val; return x; }
        char c = key.charAt(d);
        x.next[c] = put(x.next[c], key, val, d+1);
        return x;
    }
}
```
R-way trie: Java implementation (continued)

\[
\begin{align*}
public boolean contains(String key) \\
{ \text{return get(key) != null; }}
\end{align*}
\]

\[
\begin{align*}
public Value get(String key) \\
{ \text{Node x = get(root, key, 0);} \\
if (x == null) return null; \\
return (Value) x.val; \quad \text{\textcolor{red}{\textit{cast needed}}}
\end{align*}
\]

private Node get(Node x, String key, int d) 
{ 
if (x == null) return null; 
if (d == key.length()) return x; 
char c = key.charAt(d); 
return get(x.next[c], key, d+1);
}
Trie performance

Search hit. Need to examine all \( L \) characters for equality.

Search miss.
- Could have mismatch on first character.
- Typical case: examine only a few characters (sublinear).

Space. \( R \) null links at each leaf.
(but sublinear space possible if many short strings share common prefixes)

Bottom line. Fast search hit and even faster search miss, but wastes space.
Deletion in an R-way trie

To delete a key-value pair:

- Find the node corresponding to key and set value to null.
- If node has null value and all null links, remove that node (and recur).

```
delete("shells")
```
Deletion in an R-way trie

To delete a key-value pair:

- Find the node corresponding to key and set value to null.
- If node has null value and all null links, remove that node (and recur).

```
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```
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</tr>
<tr>
<td>R-way trie</td>
<td>$L$</td>
<td>$\log_R N$</td>
</tr>
</tbody>
</table>

**R-way trie.**
- Method of choice for small $R$.
- Too much memory for large $R$.

**Challenge.** Use less memory, e.g., 65,536-way trie for Unicode!
5.2 Tries

- R-way tries
- ternary search tries
- character-based operations
Ternary search tries

- Store characters and values in nodes (not keys).
- Each node has 3 children: smaller (left), equal (middle), larger (right).

Fast Algorithms for Sorting and Searching Strings

Jon L. Bentley*    Robert Sedgewick#

Abstract

We present theoretical algorithms for sorting and searching multikey data, and derive from them practical C implementations for applications in which keys are character strings. The sorting algorithm blends Quicksort and radix sort, it is competitive with the best known C sort codes. The searching algorithm blends tries and binary search trees, it is faster than hashing and other commonly used search methods. The basic ideas behind the algo-

that is competitive with the most efficient string sorting programs known. The second program is a symbol table implementation that is faster than hashing, which is commonly regarded as the fastest symbol table implementation. The symbol table implementation is much more space-efficient than multiway trees, and supports more advanced searches.

In many application programs, sorts use a Quicksort implementation based on an abstract compare operation,
Ternary search tries

- Store characters and values in nodes (not keys).
- Each node has 3 children: smaller (left), equal (middle), larger (right).

TST representation of a trie
Search hit in a TST

get("sea")

return value associated with last key character
Search miss in a TST

get("shelter")

no link to t (return null)
Ternary search trie construction demo

ternary search trie
Ternary search trie construction demo
Search in a TST

Follow links corresponding to each character in the key.
- If less, take left link; if greater, take right link.
- If equal, take the middle link and move to the next key character.

**Search hit.** Node where search ends has a non-null value.
**Search miss.** Reach a null link or node where search ends has null value.

![Diagram of TST search example](image)

- get("sea")
- mismatch: take left or right link, do not move to next char
- match: take middle link, move to next char
- return value associated with last key character
26-way trie vs. TST

26-way trie. 26 null links in each leaf.

TST. 3 null links in each leaf.
TST representation in Java

A TST node is five fields:

- A value.
- A character $c$.
- A reference to a left TST.
- A reference to a middle TST.
- A reference to a right TST.

```java
private class Node {
    private Value val;
    private char c;
    private Node left, mid, right;
}
```

Trie node representations

standard array of links (R = 26)

ternary search tree (TST)

link for keys that start with $s$

link for keys that start with $su$
public class TST\<Value\>  
{  
    private Node root;  

    private class Node  
    { /* see previous slide */  }  

    public void put(String key, Value val)  
    { root = put(root, key, val, 0);  }  

    private Node put(Node x, String key, Value val, int d)  
    {  
        char c = key.charAt(d);  
        if (x == null) { x = new Node(); x.c = c;  }  
        if (c < x.c) x.left = put(x.left, key, val, d);  
        else if (c > x.c) x.right = put(x.right, key, val, d);  
        else if (d < key.length() - 1) x.mid = put(x.mid, key, val, d+1);  
        else x.val = val;  
        return x;  
    }  
}
public boolean contains(String key)
{
    return get(key) != null;
}

public Value get(String key)
{
    Node x = get(root, key, 0);
    if (x == null) return null;
    return x.val;
}

private Node get(Node x, String key, int d)
{
    if (x == null) return null;
    char c = key.charAt(d);
    if (c < x.c) return get(x.left, key, d);
    else if (c > x.c) return get(x.right, key, d);
    else if (d < key.length() - 1) return get(x.mid, key, d+1);
    else return x;
}
## String symbol table implementation cost summary

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<tr>
<th>implementation</th>
<th>search hit</th>
<th>search miss</th>
<th>insert</th>
<th>space (references)</th>
<th>moby.txt</th>
<th>actors.txt</th>
</tr>
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<tr>
<td>red-black BST</td>
<td>$L + c \lg^2 N$</td>
<td>$c \lg^2 N$</td>
<td>$c \lg^2 N$</td>
<td>$4N$</td>
<td>1.40</td>
<td>97.4</td>
</tr>
<tr>
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<td>$L$</td>
<td>$L$</td>
<td>$L$</td>
<td>$4N$ to $16N$</td>
<td>0.76</td>
<td>40.6</td>
</tr>
<tr>
<td>R-way trie</td>
<td>$L$</td>
<td>$\log_R N$</td>
<td>$L$</td>
<td>$(R + 1)N$</td>
<td>1.12</td>
<td>out of memory</td>
</tr>
<tr>
<td>TST</td>
<td>$L + \ln N$</td>
<td>$\ln N$</td>
<td>$L + \ln N$</td>
<td>$4N$</td>
<td>0.72</td>
<td>38.7</td>
</tr>
</tbody>
</table>

**Remark.** Can build balanced TSTs via rotations to achieve $L + \log N$ worst-case guarantees.

**Bottom line.** TST is as fast as hashing (for string keys), space efficient.
Hybrid of R-way trie and TST.

- Do $R^2$-way branching at root.
- Each of $R^2$ root nodes points to a TST.

Q. What about one- and two-letter words?
## String symbol table implementation cost summary

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<td>$4 N$</td>
<td>0.72</td>
<td>38.7</td>
</tr>
<tr>
<td>TST with $R^2$</td>
<td>$L + \ln N$</td>
<td>$\ln N$</td>
<td>$L + \ln N$</td>
<td>$4 N + R^2$</td>
<td>0.51</td>
<td>32.7</td>
</tr>
</tbody>
</table>

**Bottom line.** Faster than hashing for our benchmark client.
TST vs. hashing

Hashing.
- Need to examine entire key.
- Search hits and misses cost about the same.
- Performance relies on hash function.
- Does not support ordered symbol table operations.

TSTs.
- Works only for strings (or digital keys).
- Only examines just enough key characters.
- Search miss may involve only a few characters.
- Supports ordered symbol table operations (plus others!).

Bottom line. TSTs are:
- Faster than hashing (especially for search misses).
- More flexible than red-black BSTs. [stay tuned]
5.2 Tries

- R-way tries
- ternary search tries
- character-based operations
String symbol table API

Character-based operations. The string symbol table API supports several useful character-based operations.

<table>
<thead>
<tr>
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<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>by</td>
<td>4</td>
</tr>
<tr>
<td>sea</td>
<td>6</td>
</tr>
<tr>
<td>sells</td>
<td>1</td>
</tr>
<tr>
<td>she</td>
<td>0</td>
</tr>
<tr>
<td>shells</td>
<td>3</td>
</tr>
<tr>
<td>shore</td>
<td>7</td>
</tr>
<tr>
<td>the</td>
<td>5</td>
</tr>
</tbody>
</table>

Prefix match. Keys with prefix sh: she, shells, and shore.

Wildcard match. Keys that match .he: she and the.

Longest prefix. Key that is the longest prefix of shellsort: shells.
### String symbol table API

```java
public class StringST<Value> {
    StringST() // create a symbol table with string keys
    void put(String key, Value val) // put key-value pair into the symbol table
    Value get(String key) // value paired with key
    void delete(String key) // delete key and corresponding value
    ...
    Iterable<String> keys() // all keys
    Iterable<String> keysWithPrefix(String s) // keys having s as a prefix
    Iterable<String> keysThatMatch(String s) // keys that match s (where . is a wildcard)
    String longestPrefixOf(String s) // longest key that is a prefix of s
}
```

**Remark.** Can also add other ordered ST methods, e.g., `floor()` and `rank()`. 
Warmup: ordered iteration

To iterate through all keys in sorted order:

- Do inorder traversal of trie; add keys encountered to a queue.
- Maintain sequence of characters on path from root to node.

```
keys()
  key  q
    b
    by
    s
    se
  sea by sea
  sell
  sells by sea sells
    sh
    she by sea sells she
  shell
  shells by sea sells she shells
    sho
    shor
  shore by sea sells she shells shore
    t
    th
  the by sea sells she shells shore the
```
Ordered iteration: Java implementation

To iterate through all keys in sorted order:

- Do inorder traversal of trie; add keys encountered to a queue.
- Maintain sequence of characters on path from root to node.

```java
public Iterable<String> keys() {
    Queue<String> queue = new Queue<String>();
    collect(root, "", queue);
    return queue;
}

private void collect(Node x, String prefix, Queue<String> q) {
    if (x == null) return;
    if (x.val != null) q.enqueue(prefix);
    for (char c = 0; c < R; c++)
        collect(x.next[c], prefix + c, q);
}
```
Prefix matches

Find all keys in a symbol table starting with a given prefix.

Ex. Autocomplete in a cell phone, search bar, text editor, or shell.
- User types characters one at a time.
- System reports all matching strings.
Prefix matches in an R-way trie

Find all keys in a symbol table starting with a given prefix.

```java
public Iterable<String> keysWithPrefix(String prefix) {
    Queue<String> queue = new Queue<String>();
    Node x = get(root, prefix, 0);
    collect(x, prefix, queue);
    return queue;
}
```

```
key   queue
sh    she
she   shells
shell shells
shells sho
shor
shore  she shells shore
```
Longest prefix

Find longest key in symbol table that is a prefix of query string.

Ex. To send packet toward destination IP address, router chooses IP address in routing table that is longest prefix match.

"128"
"128.112"
"128.112.055"
"128.112.055.15"
"128.112.136"
"128.112.155.11"
"128.112.155.13"
"128.222"
"128.222.136"

Note. Not the same as floor: \( \text{floor}("128.112.100.16") = "128.112.055.15"\)
Longest prefix in an R-way trie

Find longest key in symbol table that is a prefix of query string.
- Search for query string.
- Keep track of longest key encountered.

Possibilities for longestPrefixOf()
Longest prefix in an R-way trie: Java implementation

Find longest key in symbol table that is a prefix of query string.

- Search for query string.
- Keep track of longest key encountered.

```java
public String longestPrefixOf(String query) {
    int length = search(root, query, 0, 0);
    return query.substring(0, length);
}

private int search(Node x, String query, int d, int length) {
    if (x == null) return length;
    if (x.val != null) length = d;
    if (d == query.length()) return length;
    char c = query.charAt(d);
    return search(x.next[c], query, d+1, length);
}
```
T9 texting

**Goal.** Type text messages on a phone keypad.

**Multi-tap input.** Enter a letter by repeatedly pressing a key until the desired letter appears.

"a much faster and more fun way to enter text"

**T9 text input.**
- Find all words that correspond to given sequence of numbers.
- Press 0 to see all completion options.

**Ex.** hello
- Multi-tap: 4 4 3 3 5 5 5 5 5 6 6 6
- T9: 4 3 5 5 6

**Q.** How to implement?

www.t9.com
Patricia trie

**Patricia trie.** [Practical Algorithm to Retrieve Information Coded in Alphanumeric]

- Remove one-way branching.
- Each node represents a sequence of characters.
- Implementation: one step beyond this course.

**Applications.**

- Database search.
- P2P network search.
- IP routing tables: find longest prefix match.
- Compressed quad-tree for N-body simulation.
- Efficiently storing and querying XML documents.

Also known as: crit-bit tree, radix tree.
Suffix tree

Suffix tree.

- Patricia trie of suffixes of a string.
- Linear-time construction: beyond this course.

Applications.

- Linear-time: longest repeated substring, longest common substring, longest palindromic substring, substring search, tandem repeats, ....
- Computational biology databases (BLAST, FASTA).
String symbol tables summary

A success story in algorithm design and analysis.

Red-black BST.
- Performance guarantee: $\log N$ key compares.
- Supports ordered symbol table API.

Hash tables.
- Performance guarantee: constant number of probes.
- Requires good hash function for key type.

Tries. R-way, TST.
- Performance guarantee: $\log N$ characters accessed.
- Supports character-based operations.

Bottom line. You can get at anything by examining 50-100 bits (!!!)