1.4 Analysis of Algorithms

- introduction
- observations
- mathematical models
- order-of-growth classifications
- theory of algorithms
- memory

Cast of characters

Programmer needs to develop a working solution.

Client wants to solve problem efficiently.

Student might play any or all of these roles someday.

Theoretician wants to understand.

Basic blocking and tackling is sometimes necessary. [this lecture]

Running time

“As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise—By what course of calculation can these results be arrived at by the machine in the shortest time?” — Charles Babbage (1864)
Reasons to analyze algorithms

- Predict performance.
- Compare algorithms.
- Provide guarantees.
- Understand theoretical basis.

Primary practical reason: avoid performance bugs.

Some algorithmic successes

**Discrete Fourier transform.**
- Break down waveform of $N$ samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ....
- Brute force: $N^2$ steps.
- FFT algorithm: $N \log N$ steps, enables new technology.

**N-body simulation.**
- Simulate gravitational interactions among $N$ bodies.
- Brute force: $N^2$ steps.
- Barnes-Hut algorithm: $N \log N$ steps, enables new research.

The challenge

Q. Will my program be able to solve a large practical input?

**Insight.** [Knuth 1970s] Use scientific method to understand performance.
Scientific method applied to analysis of algorithms

A framework for predicting performance and comparing algorithms.

Scientific method.
- Observe some feature of the natural world.
- Hypothesize a model that is consistent with the observations.
- Predict events using the hypothesis.
- Verify the predictions by making further observations.
- Validate by repeating until the hypothesis and observations agree.

Principles.
- Experiments must be reproducible.
- Hypotheses must be falsifiable.

Feature of the natural world. Computer itself.

Example: 3-SUM

3-SUM. Given $N$ distinct integers, how many triples sum to exactly zero?

```java
public class ThreeSum {
    public static int count(int[] a) {
        int N = a.length;
        int count = 0;
        for (int i = 0; i < N; i++)
            for (int j = i+1; j < N; j++)
                for (int k = j+1; k < N; k++)
                    if (a[i] + a[j] + a[k] == 0)
                       count++;
        return count;
    }

    public static void main(String[] args) {
        int[] a = In.readInts(args[0]);
        StdOut.println(count(a));
    }
}
```

Context. Deeply related to problems in computational geometry.

1.4 Analysis of Algorithms

- introduction
- observations
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- order-of-growth classifications
- theory of algorithms
- memory

3-SUM: brute-force algorithm

Check each triple for simplicity, ignore integer overflow.
Measuring the running time

Q. How to time a program?
A. Manual.

<table>
<thead>
<tr>
<th>N</th>
<th>time (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>250</td>
<td>0.0</td>
</tr>
<tr>
<td>500</td>
<td>0.0</td>
</tr>
<tr>
<td>1,000</td>
<td>0.1</td>
</tr>
<tr>
<td>2,000</td>
<td>0.8</td>
</tr>
<tr>
<td>4,000</td>
<td>6.4</td>
</tr>
<tr>
<td>8,000</td>
<td>51.1</td>
</tr>
<tr>
<td>16,000</td>
<td>?</td>
</tr>
</tbody>
</table>

Empirical analysis

Run the program for various input sizes and measure running time.
Data analysis

**Standard plot.** Plot running time $T(N)$ vs. input size $N$.

![Standard plot](image)

**Log-log plot.** Plot running time $T(N)$ vs. input size $N$ using log-log scale.

![Log-log plot](image)

$\log(T(N)) = b \log N + c$

$b = 2.999$

$c = -33.2103$

$T(N) = a N^b$, where $a = 2^c$

**Regression.** Fit straight line through data points: $a N^b$.

Hypothesis. The running time is about $1.006 \times 10^{-10} \times N^{2.999}$ seconds.

Prediction and validation

**Hypothesis.** The running time is about $1.006 \times 10^{-10} \times N^{2.999}$ seconds.

**Predictions.**
- 51.0 seconds for $N = 8,000$.
- 408.1 seconds for $N = 16,000$.

**Observations.**

<table>
<thead>
<tr>
<th>$N$</th>
<th>time (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8,000</td>
<td>51.1</td>
</tr>
<tr>
<td>8,000</td>
<td>51.0</td>
</tr>
<tr>
<td>8,000</td>
<td>51.1</td>
</tr>
<tr>
<td>16,000</td>
<td>410.8</td>
</tr>
</tbody>
</table>

**Doubling hypothesis.** Quick way to estimate $b$ in a power-law relationship.

Run program, **doubling** the size of the input.

<table>
<thead>
<tr>
<th>$N$</th>
<th>time (seconds)</th>
<th>ratio</th>
<th>$\log$ ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>250</td>
<td>0.0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>500</td>
<td>0.0</td>
<td>4.8</td>
<td>2.3</td>
</tr>
<tr>
<td>1,000</td>
<td>0.1</td>
<td>6.9</td>
<td>2.8</td>
</tr>
<tr>
<td>2,000</td>
<td>0.8</td>
<td>7.7</td>
<td>2.9</td>
</tr>
<tr>
<td>4,000</td>
<td>6.4</td>
<td>8.0</td>
<td>3.0</td>
</tr>
<tr>
<td>8,000</td>
<td>51.1</td>
<td>8.0</td>
<td>3.0</td>
</tr>
</tbody>
</table>

seems to converge to a constant $b \approx 3$

**Hypothesis.** Running time is about $a N^b$ with $b = \log$ ratio.

**Caveat.** Cannot identify logarithmic factors with doubling hypothesis.
Doubling hypothesis

Doubling hypothesis. Quick way to estimate $b$ in a power-law relationship.

Q. How to estimate $a$ (assuming we know $b$)?
A. Run the program (for a sufficient large value of $N$) and solve for $a$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>time (seconds) $t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8,000</td>
<td>51.1</td>
</tr>
<tr>
<td>8,000</td>
<td>51.0</td>
</tr>
<tr>
<td>8,000</td>
<td>51.1</td>
</tr>
</tbody>
</table>

$51.1 = a \times 8000^3$

$\Rightarrow a = 0.998 \times 10^{-10}$

Hypothesis. Running time is about $0.998 \times 10^{-10} \times N^3$ seconds.

Experimental algorithmics

System independent effects.
- Algorithm.
- Input data.

determines exponent $b$ in power law

determines constant $a$ in power law

System dependent effects.
- Hardware: CPU, memory, cache, ...
- Software: compiler, interpreter, garbage collector, ...
- System: operating system, network, other apps, ...

Bad news. Difficult to get precise measurements.
Good news. Much easier and cheaper than other sciences.

e.g., can run huge number of experiments

Mathematical models for running time

Total running time: sum of cost $\times$ frequency for all operations.
- Need to analyze program to determine set of operations.
- Cost depends on machine, compiler.
- Frequency depends on algorithm, input data.

In principle, accurate mathematical models are available.
### Cost of basic operations

<table>
<thead>
<tr>
<th>operation</th>
<th>example</th>
<th>nanoseconds</th>
</tr>
</thead>
<tbody>
<tr>
<td>integer add</td>
<td>a + b</td>
<td>2.1</td>
</tr>
<tr>
<td>integer multiply</td>
<td>a * b</td>
<td>2.4</td>
</tr>
<tr>
<td>integer divide</td>
<td>a / b</td>
<td>5.4</td>
</tr>
<tr>
<td>floating-point add</td>
<td>a + b</td>
<td>4.6</td>
</tr>
<tr>
<td>floating-point multiply</td>
<td>a * b</td>
<td>4.2</td>
</tr>
<tr>
<td>floating-point divide</td>
<td>a / b</td>
<td>13.5</td>
</tr>
<tr>
<td>sine</td>
<td>Math.sin(theta)</td>
<td>91.3</td>
</tr>
<tr>
<td>arctangent</td>
<td>Math.atan2(y, x)</td>
<td>129.0</td>
</tr>
</tbody>
</table>

† Running OS X on Macbook Pro 2.2GHz with 2GB RAM

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### Example: 1-SUM

**Q.** How many instructions as a function of input size $N$?

```
int count = 0;
for (int i = 0; i < N; i++)
  if (a[i] == 0)
    count++;
```

<table>
<thead>
<tr>
<th>operation</th>
<th>frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>variable declaration</td>
<td>2</td>
</tr>
<tr>
<td>assignment statement</td>
<td>2</td>
</tr>
<tr>
<td>less than compare</td>
<td>N + 1</td>
</tr>
<tr>
<td>equal to compare</td>
<td>N</td>
</tr>
<tr>
<td>array access</td>
<td>N</td>
</tr>
<tr>
<td>increment</td>
<td>N to 2N</td>
</tr>
</tbody>
</table>

---

### Example: 2-SUM

**Q.** How many instructions as a function of input size $N$?

```
int count = 0;
for (int i = 0; i < N; i++)
  for (int j = i+1; j < N; j++)
    if (a[i] + a[j] == 0)
      count++;
```

$$0 + 1 + 2 + \ldots + (N-1) = \frac{1}{2} N (N - 1)$$

<table>
<thead>
<tr>
<th>operation</th>
<th>frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>variable declaration</td>
<td>N + 2</td>
</tr>
<tr>
<td>assignment statement</td>
<td>N + 2</td>
</tr>
<tr>
<td>less than compare</td>
<td>½ (N + 1) (N + 2)</td>
</tr>
<tr>
<td>equal to compare</td>
<td>½ N (N - 1)</td>
</tr>
<tr>
<td>array access</td>
<td>N (N - 1)</td>
</tr>
<tr>
<td>increment</td>
<td>½ N (N - 1) to N (N - 1)</td>
</tr>
</tbody>
</table>

Novice mistake. Abusive string concatenation.
Simplifying the calculations

“It is convenient to have a measure of the amount of work involved in a computing process, even though it be a very crude one. We may count up the number of times that various elementary operations are applied in the whole process and then give them various weights. We might, for instance, count the number of additions, subtractions, multiplications, divisions, recording of numbers, and extractions of figures from tables. In the case of computing with matrices most of the work consists of multiplications and writing down numbers, and we shall therefore only attempt to count the number of multiplications and recordings.” — Alan Turing

Simplification 1: cost model

Cost model. Use some basic operation as a proxy for running time.

```
int count = 0;
for (int i = 0; i < N; i++)
    for (int j = i+1; j < N; j++)
        if (a[i] + a[j] == 0)
            count++;
```

\[
0 + 1 + 2 + \ldots + (N - 1) = \frac{1}{2} N (N - 1) = \binom{N}{2}
\]

<table>
<thead>
<tr>
<th>operation</th>
<th>frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>variable declaration</td>
<td>(N + 2)</td>
</tr>
<tr>
<td>assignment statement</td>
<td>(N + 2)</td>
</tr>
<tr>
<td>less than compare</td>
<td>(\frac{1}{2} (N + 1) (N + 2))</td>
</tr>
<tr>
<td>equal to compare</td>
<td>(\frac{1}{2} N (N - 1))</td>
</tr>
<tr>
<td>array access</td>
<td>(N (N - 1))</td>
</tr>
<tr>
<td>increment</td>
<td>(\frac{1}{2} N (N - 1)) to (N (N - 1))</td>
</tr>
</tbody>
</table>

Cost model = array accesses

Simplification 2: tilded notation

- Estimate running time (or memory) as a function of input size \(N\).
- Ignore lower order terms.
  - when \(N\) is large, terms are negligible
  - when \(N\) is small, we don’t care

Ex 1. \(\frac{1}{6} N^3 + 20 N + 16 \approx \frac{1}{6} N^3\)
Ex 2. \(\frac{1}{6} N^3 + 100 N^{4/3} + 56 \approx \frac{1}{6} N^3\)
Ex 3. \(\frac{1}{6} N^3 - \frac{1}{2} N^2 + \frac{1}{3} N \approx \frac{1}{6} N^3\)

discard lower-order terms (e.g., \(N = 1000\): 500 thousand vs. 166 million)

Technical definition. \(f(N) \sim g(N)\) means \(\lim_{N \to \infty} \frac{f(N)}{g(N)} = 1\)
Example: 2-SUM

Q. Approximately how many array accesses as a function of input size $N$?

```c
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0)
         count++;
```

A. $\sim N^2$ array accesses.

Bottom line. Use cost model and tilde notation to simplify counts.

Example: 3-SUM

Q. Approximately how many array accesses as a function of input size $N$?

```c
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      for (int k = j+1; k < N; k++)
         if (a[i] + a[j] + a[k] == 0)
            count++;
```

A. $\sim \frac{1}{2} N^3$ array accesses.

Bottom line. Use cost model and tilde notation to simplify counts.

---

Estimating a discrete sum

Q. How to estimate a discrete sum?

A1. Take discrete mathematics course.

A2. Replace the sum with an integral, and use calculus!

**Ex 1.** $1 + 2 + \ldots + N.$

\[
\sum_{i=1}^{N} i \sim \int_{x=1}^{N} x \, dx \sim \frac{1}{2} N^2
\]

**Ex 2.** $1^k + 2^k + \ldots + N^k.$

\[
\sum_{i=1}^{N} i^k \sim \int_{x=1}^{N} x^k \, dx \sim \frac{1}{k+1} N^{k+1}
\]

**Ex 3.** $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N}.$

\[
\sum_{i=1}^{N} \frac{1}{i} \sim \int_{x=1}^{N} \frac{1}{x} \, dx = \ln N
\]

**Ex 4.** 3-sum triple loop.

\[
\sum_{i=1}^{N} \sum_{j=i+1}^{N} \sum_{k=j+1}^{N} 1 \sim \int_{x=1}^{N} \int_{y=x}^{N} \int_{z=y}^{N} dz \, dy \, dx \sim \frac{1}{6} N^3
\]

Caveat. Integral trick doesn’t always work!
Estimating a discrete sum

Q. How to estimate a discrete sum?
A3. Use Maple or Wolfram Alpha.

Mathematical models for running time

In principle, accurate mathematical models are available.

In practice,
- Formulas can be complicated.
- Advanced mathematics might be required.
- Exact models best left for experts.

Bottom line. We use approximate models in this course: $T(N) \sim cN^3$.

Common order-of-growth classifications

Good news. the small set of functions
$1, \log N, N, \log(N)N, N^2, N^3,$ and $2^N$
suffices to describe order-of-growth of typical algorithms.
### Binary search demo

**Goal.** Given a sorted array and a key, find index of the key in the array?

**Binary search.** Compare key against middle entry.
- Too small, go left.
- Too big, go right.
- Equal, found.

**Successful search for 33**

<table>
<thead>
<tr>
<th>6</th>
<th>13</th>
<th>14</th>
<th>25</th>
<th>33</th>
<th>43</th>
<th>51</th>
<th>53</th>
<th>64</th>
<th>72</th>
<th>84</th>
<th>93</th>
<th>95</th>
<th>96</th>
<th>97</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
</tr>
</tbody>
</table>

Successful search for 33.

### Binary search: Java implementation

**Trivial to implement?**
- First binary search published in 1946; first bug-free one in 1962.
- Bug in Java’s Arrays.binarySearch() discovered in 2006.

```java
public static int binarySearch(int[] a, int key) {
    int lo = 0, hi = a.length - 1;
    while (lo <= hi) {
        int mid = lo + (hi - lo) / 2;
        if (key < a[mid]) hi = mid - 1;
        else if (key > a[mid]) lo = mid + 1;
        else return mid;
    }
    return -1;
}
```

**Invariant.** If key appears in the array `a[]`, then `a[lo] ≤ key ≤ a[hi].`

### Binary search: mathematical analysis

**Proposition.** Binary search uses at most `1 + \log N` key compares to search in a sorted array of size `N`.

**Def.** \( T(N) = \# \) key compares to binary search a sorted subarray of size \( \leq N \).

**Binary search recurrence.** \( T(N) \leq T(N/2) + 1 \) for \( N > 1 \), with \( T(1) = 1 \).

**Pf sketch.**

\[
\begin{align*}
T(N) & \leq T(N/2) + 1 & \text{given} \\
& \leq T(N/4) + 1 + 1 & \text{apply recurrence to first term} \\
& \leq T(N/8) + 1 + 1 + 1 & \text{apply recurrence to first term} \\
& \ldots \\
& \leq T(N/N) + 1 + 1 + \ldots + 1 & \text{stop applying, } T(1) = 1 \\
& = 1 + \log N 
\end{align*}
\]
An $N^2 \log N$ algorithm for 3-SUM

**Sorting-based algorithm.**
- Step 1: Sort the $N$ (distinct) numbers.
- Step 2: For each pair of numbers $a[i]$ and $a[j]$, binary search for $-(a[i] + a[j])$.

**Analysis.** Order of growth is $N^2 \log N$.
- Step 1: $N^2$ with insertion sort.
- Step 2: $N^2 \log N$ with binary search.

**Remark.** Can achieve $N^2$ by modifying binary search step.

<table>
<thead>
<tr>
<th>input</th>
<th>30</th>
<th>-40</th>
<th>-20</th>
<th>-10</th>
<th>40</th>
<th>0</th>
<th>10</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>sort</td>
<td>-40</td>
<td>-20</td>
<td>-10</td>
<td>0</td>
<td>5</td>
<td>10</td>
<td>30</td>
<td>40</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>binary search</th>
<th>60</th>
<th>50</th>
<th>40</th>
<th>30</th>
<th>20</th>
<th>10</th>
<th>0</th>
<th></th>
</tr>
</thead>
</table>

Comparing programs

**Hypothesis.** The sorting-based $N^2 \log N$ algorithm for 3-SUM is significantly faster in practice than the brute-force $N^3$ algorithm.

![ThreeSum.java](image)

Guiding principle. Typically, better order of growth $\Rightarrow$ faster in practice.

Types of analyses

**Best case.** Lower bound on cost.
- Determined by “easiest” input.
- Provides a goal for all inputs.

**Worst case.** Upper bound on cost.
- Determined by “most difficult” input.
- Provides a guarantee for all inputs.

**Average case.** Expected cost for random input.
- Need a model for “random” input.
- Provides a way to predict performance.

**Ex 1.** Array accesses for brute-force 3-SUM.
- Best: $\sim \frac{1}{3} N^3$
- Average: $\sim \frac{1}{3} N^3$
- Worst: $\sim \frac{1}{3} N^3$

**Ex 2.** Compares for binary search.
- Best: $\sim 1$
- Average: $\sim \log N$
- Worst: $\sim \log N$
Types of analyses

Best case. Lower bound on cost.
Worst case. Upper bound on cost.
Average case. “Expected” cost.

Actual data might not match input model?
• Need to understand input to effectively process it.
• Approach 1: design for the worst case.
• Approach 2: randomize, depend on probabilistic guarantee.

Theory of algorithms

Goals.
• Establish “difficulty” of a problem.
• Develop “optimal” algorithms.

Approach.
• Suppress details in analysis: analyze “to within a constant factor”.
• Eliminate variability in input model by focusing on the worst case.

Optimal algorithm.
• Performance guarantee (to within a constant factor) for any input.
• No algorithm can provide a better performance guarantee.

Commonly-used notations in the theory of algorithms

<table>
<thead>
<tr>
<th>notation</th>
<th>provides</th>
<th>example</th>
<th>shorthand for</th>
<th>used to</th>
</tr>
</thead>
<tbody>
<tr>
<td>Big Theta</td>
<td>asymptotic</td>
<td>( \Theta(N^2) )</td>
<td>½ ( N^2 ) 10 ( N^2 ) 5 ( N^2 + 22 N \log N + 3N )</td>
<td>classify algorithms</td>
</tr>
<tr>
<td></td>
<td>order of growth</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Big Oh</td>
<td>( \Theta(N^2) ) and smaller</td>
<td>( O(N^2) )</td>
<td>10 ( N^2 ) 100 ( N ) 22 ( N \log N + 3N )</td>
<td>develop upper bounds</td>
</tr>
<tr>
<td>Big Omega</td>
<td>( \Theta(N^2) ) and larger</td>
<td>( \Omega(N^2) )</td>
<td>½ ( N^2 ) ( N^5 ) ( N^3 + 22 N \log N + 3N )</td>
<td>develop lower bounds</td>
</tr>
</tbody>
</table>

Theory of algorithms: example 1

Goals.
• Establish “difficulty” of a problem and develop “optimal” algorithms.
• Ex. 1-SUM = “Is there a 0 in the array?”

Upper bound. A specific algorithm.
• Ex. Brute-force algorithm for 1-SUM: Look at every array entry.
• Running time of the optimal algorithm for 1-SUM is \( O(N) \).

Lower bound. Proof that no algorithm can do better.
• Ex. Have to examine all \( N \) entries (any unexamined one might be 0).
• Running time of the optimal algorithm for 1-SUM is \( \Omega(N) \).

Optimal algorithm.
• Lower bound equals upper bound (to within a constant factor).
• Ex. Brute-force algorithm for 1-SUM is optimal: its running time is \( \Theta(N) \).
Theory of algorithms: example 2

Goals.
- Establish “difficulty” of a problem and develop “optimal” algorithms.
- Ex. 3-SUM.

Upper bound. A specific algorithm.
- Ex. Brute-force algorithm for 3-SUM.
- Running time of the optimal algorithm for 3-SUM is \( O(N^3) \).

Algorithm design approach

Start.
- Develop an algorithm.
- Prove a lower bound.

Gap?
- Lower the upper bound (discover a new algorithm).
- Raise the lower bound (more difficult).

Golden Age of Algorithm Design.
- 1970s–.
- Steadily decreasing upper bounds for many important problems.
- Many known optimal algorithms.

Caveats.
- Overly pessimistic to focus on worst case?
- Need better than “to within a constant factor” to predict performance.

Commonly-used notations

<table>
<thead>
<tr>
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<th>shorthand for</th>
<th>used to</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tilde</td>
<td>leading term</td>
<td>( \sim 10N^2 )</td>
<td>( 10 N^2 ) ( 10 N^2 + 22 N \log N ) ( 10 N^2 + 2 N + 37 )</td>
<td>provide approximate model</td>
</tr>
<tr>
<td>Big Theta</td>
<td>asymptotic growth rate</td>
<td>( O(N^2) )</td>
<td>( \frac{1}{2} N^2 ) ( 10 N^2 ) ( 5 N^2 + 22 N \log N + 3N )</td>
<td>classify algorithms</td>
</tr>
<tr>
<td>Big Oh</td>
<td>( \Theta(N^2) ) and smaller</td>
<td>( O(N^2) )</td>
<td>( 10 N^2 ) ( 100 N ) ( 22 N \log N + 3N )</td>
<td>develop upper bounds</td>
</tr>
<tr>
<td>Big Omega</td>
<td>( \Omega(N^2) ) and larger</td>
<td>( O(N^2) )</td>
<td>( \frac{1}{2} N^2 ) ( N^5 ) ( N^3 + 22 N \log N + 3N )</td>
<td>develop lower bounds</td>
</tr>
</tbody>
</table>

Common mistake. Interpreting big-Oh as an approximate model.
This course. Focus on approximate models: use Tilde-notations.
1.4 Analysis of Algorithms

- introduction
- observations
- mathematical models
- order-of-growth classifications
- theory of algorithms
- memory

### Basics

**Bit.** 0 or 1.

**Byte.** 8 bits.

**Megabyte (MB).** 1 million or $2^{20}$ bytes.

**Gigabyte (GB).** 1 billion or $2^{30}$ bytes.

**64-bit machine.** We assume a 64-bit machine with 8 byte pointers.

- Can address more memory.
- Pointers use more space.

Some JVMs “compress” ordinary object pointers to 4 bytes to avoid this cost.

### Typical memory usage for primitive types and arrays

<table>
<thead>
<tr>
<th>type</th>
<th>bytes</th>
</tr>
</thead>
<tbody>
<tr>
<td>boolean</td>
<td>1</td>
</tr>
<tr>
<td>byte</td>
<td>1</td>
</tr>
<tr>
<td>char</td>
<td>2</td>
</tr>
<tr>
<td>int</td>
<td>4</td>
</tr>
<tr>
<td>float</td>
<td>4</td>
</tr>
<tr>
<td>long</td>
<td>8</td>
</tr>
<tr>
<td>double</td>
<td>8</td>
</tr>
</tbody>
</table>

**for primitive types**

<table>
<thead>
<tr>
<th>type</th>
<th>bytes</th>
</tr>
</thead>
<tbody>
<tr>
<td>char[]</td>
<td>$2N + 24$</td>
</tr>
<tr>
<td>int[]</td>
<td>$4N + 24$</td>
</tr>
<tr>
<td>double[]</td>
<td>$8N + 24$</td>
</tr>
</tbody>
</table>

**for one-dimensional arrays**

**for two-dimensional arrays**

### Typical memory usage for objects in Java

**Object overhead.** 16 bytes.

**Reference.** 8 bytes.

**Padding.** Each object uses a multiple of 8 bytes.

**Ex 1.** A Date object uses 32 bytes of memory.

```java
public class Date {
    private int day;
    private int month;
    private int year;

    ...}
```

16 bytes (object overhead)

4 bytes (int)
4 bytes (int)
4 bytes (int)
4 bytes (padding)
32 bytes
Typical memory usage summary

Total memory usage for a data type value:
- Primitive type: 4 bytes for int, 8 bytes for double, ...
- Object reference: 8 bytes.
- Array: 24 bytes + memory for each array entry.
- Object: 16 bytes + memory for each instance variable + 8 bytes if inner class (for pointer to enclosing class).
- Padding: round up to multiple of 8 bytes.

Shallow memory usage: Don’t count referenced objects.

Deep memory usage: If array entry or instance variable is a reference, add memory (recursively) for referenced object.

Example

Q. How much memory does WeightedQuickUnionUF use as a function of N?
Use tilde notation to simplify your answer.

A. \( 8N + 88 \sim 8N \) bytes.

Turning the crank: summary

Empirical analysis.
- Execute program to perform experiments.
- Assume power law and formulate a hypothesis for running time.
- Model enables us to make predictions.

Mathematical analysis.
- Analyze algorithm to count frequency of operations.
- Use tilde notation to simplify analysis.
- Model enables us to explain behavior.

Scientific method.
- Mathematical model is independent of a particular system; applies to machines not yet built.
- Empirical analysis is necessary to validate mathematical models and to make predictions.