



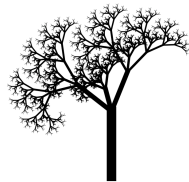
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### Overview

**What is recursion?** When one function calls **itself** directly or indirectly.

#### Why learn recursion?

- New mode of thinking.
- Powerful programming paradigm.



**Many computations are naturally self-referential.**

- Binary search, mergesort, FFT, **GCD**.
- Linked data structures.
- A folder contains files and other folders.



M. C. Escher, 1956

**Closely related to mathematical induction.**

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## 2.3 Recursion



### Mathematical Induction

**Mathematical induction.** Prove a statement involving an integer  $N$  by

- **base case:** Prove it for some specific  $N$  (usually 0 or 1).
- **induction step:** Assume it to be true for all positive integers less than  $N$ , use that fact to prove it for  $N$ .

**Ex.** Sum of the first  $N$  odd integers is  $N^2$ .

1	=	1
1 + 3	=	4
1 + 3 + 5	=	9
1 + 3 + 5 + 7	=	16
1 + 3 + 5 + 7 + 9	=	25
...		...

Base case: True for  $N = 1$ .

Induction step:

- Let  $T(N)$  be the sum of the first  $N$  odd integers:  $1 + 3 + 5 + \dots + (2N - 1)$ .
- Assume that  $T(N-1) = (N-1)^2$ .
- $T(N) = T(N-1) + (2N - 1)$   
 $= (N-1)^2 + (2N - 1)$   
 $= N^2 - 2N + 1 + (2N - 1)$   
 $= N^2$

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## Recursive Program

**Recursive Program.** Implement a function having integer arguments by

- **base case:** Do something specific in response to "base" argument values.
- **reduction step:** Assume the function works for the base case, and use the function to implement **itself** for general argument values.

```
public static String convert(int x)
{
    if (x == 1) return "1";
    return convert(x/2) + (x % 2);
}
```

automatic cast to  
String  
(either "0" or "1")

**Ex 1.** Convert positive int to binary string.

Base case: return "1" for x = 1.

Reduction step:

- convert  $x/2$  to binary
- append "0" if x even
- append "1" if x odd

37            18  
"100101" = "10010" + "1"

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Use **mathematical induction** to prove recursive programs correct

**Recursive Program.** Implement a function having integer arguments by

- base case: Implementing it for some specific values of the arguments.
- reduction step: Assume the function works for smaller values of its arguments and use it to implement it for the given values.

**Mathematical induction.** Prove a statement involving an integer N by

- base case: Prove it for some specific N (usually 0 or 1).
- induction step: Assume it to be true for all positive integers less than N, use that fact to prove it for N.

**Ex.** Proof by induction for convert().

Base case: returns "1" for x = 1.

**Induction step:**

- Assume correct for  $x/2$ .
- correct to append "0" if x even since  $x = 2(x/2)$
- correct to append "1" if x odd since  $x = 2(x/2) + 1$

```
public static String convert(int x)
{
    if (x == 1) return "1";
    return convert(x/2) + (x % 2);
}
```

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## Recursive Program

**Recursive Program.** Implement a function having integer arguments by

- **base case:** Implementing it for some specific values of the arguments.
- **reduction step:** Assume the function works for smaller values of its arguments and use it to implement it for the given values.

```
public class Binary
{
    public static int convert(int x)
    {
        if (x == 1) return "1";
        else return convert(x/2) + (x % 2);
    }

    public static void main(String[] args)
    {
        int x = Integer.parseInt(args[0]);
        System.out.println(convert(x));
    }
}
```

```
% java Binary 6
110
% java Binary 37
100101
% java Binary 999999
11110100001000111111
```

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## Recursion vs. Iteration

Every program with 1 recursive call corresponds to a loop.

```
public static String convert(int x)
{
    if (x == 1) return "1";
    return convert(x/2) + (x % 2);
}
```

```
public static String convertNR(int x)
{
    String s = "1";
    while (x > 1)
    {
        s = (x % 2) + s;
        x = x/2;
    }
    return s;
}
```

**Reasons to use recursion:**

- code more compact
- easier to understand
- easier to reason about correctness
- easy to add multiple recursive calls (stay tuned)

**Reasons not to use recursion:** (stay tuned)

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## Greatest Common Divisor

**Gcd.** Find largest integer that evenly divides into p and q.

**Ex.**  $\text{gcd}(4032, 1272) = 24$ .

$$\begin{aligned} 4032 &= 2^6 \times 3^2 \times 7^1 \\ 1272 &= 2^3 \times 3^1 \times 53^1 \\ \text{gcd} &= 2^3 \times 3^1 = 24 \end{aligned}$$

### Applications.

- Simplify fractions:  $1272/4032 = 53/168$ .
- RSA cryptosystem.

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## Greatest Common Divisor

**GCD.** Find largest integer that evenly divides into p and q.

**Euclid's algorithm.** [Euclid 300 BCE]

$$\text{gcd}(p, q) = \begin{cases} p & \text{if } q = 0 \\ \text{gcd}(q, p \% q) & \text{otherwise} \end{cases}$$

← base case  
← reduction step, converges to base case

$$\begin{aligned} \text{gcd}(4032, 1272) &= \text{gcd}(1272, 216) \\ &= \text{gcd}(216, 192) \\ &= \text{gcd}(192, 24) \\ &= \text{gcd}(24, 0) \\ &= 24. \end{aligned}$$

$4032 = 3 \times 1272 + 216$

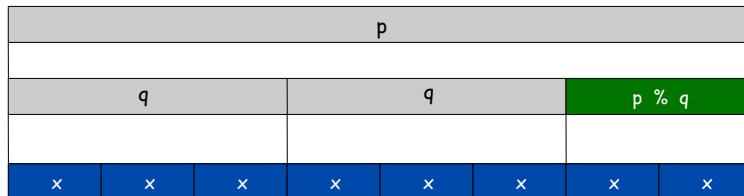
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## Euclid's Algorithm

**GCD.** Find largest integer d that evenly divides into p and q.

$$\text{gcd}(p, q) = \begin{cases} p & \text{if } q = 0 \\ \text{gcd}(q, p \% q) & \text{otherwise} \end{cases}$$

← base case  
← reduction step, converges to base case



$$\begin{aligned} p &= 8x \\ q &= 3x \end{aligned}$$

$$\text{gcd}(p, q) = \text{gcd}(3x, 2x) = x$$

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## Euclid's Algorithm

**GCD.** Find largest integer d that evenly divides into p and q.

$$\text{gcd}(p, q) = \begin{cases} p & \text{if } q = 0 \\ \text{gcd}(q, p \% q) & \text{otherwise} \end{cases}$$

← base case  
← reduction step, converges to base case

### Recursive program

```
public static int gcd(int p, int q)
{
    if (q == 0) return p;
    else return gcd(q, p % q);
}
```

← base case  
← reduction step

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## Possible debugging challenges with recursion

### Missing base case.

```
public static double BAD(int N)
{
    return BAD(N-1) + 1.0/N;
}
```

### No convergence guarantee.

```
public static double BAD(int N)
{
    if (N == 1) return 1.0;
    return BAD(1 + N/2) + 1.0/N;
}
```

Both lead to INFINITE RECURSIVE LOOP (bad news).

Try it!

← so that you can recognize and deal with it if it later happens to you

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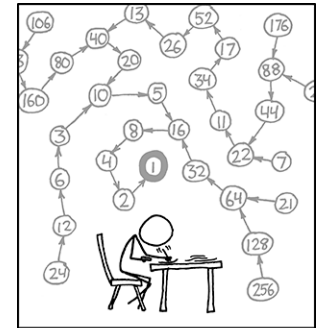
## Collatz Sequence

### Collatz sequence.

- If n is 1, stop.
- If n is even, divide by 2.
- If n is odd, multiply by 3 and add 1.

Ex. 35 106 53 160 80 40 20 10 5 16 8 4 2 1.

```
public static void collatz(int N)
{
    Stdout.print(N + " ");
    if (N == 1) return;
    if (N % 2 == 0) collatz(N / 2);
    collatz(3*N + 1);
}
```



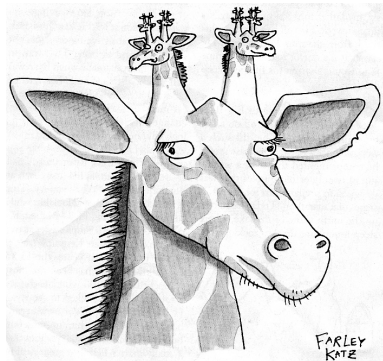
THE COLLATZ CONJECTURE STATES THAT IF YOU PICK A NUMBER, AND IF IT'S EVEN DIVIDE IT BY TWO AND IF IT'S ODD MULTIPLY IT BY THREE AND ADD ONE, AND YOU REPEAT THIS PROCEDURE LONG ENOUGH, EVENTUALLY YOUR FRIENDS WILL STOP CALLING TO SEE IF YOU WANT TO HANG OUT.

No one knows whether or not this function terminates for all N (!)

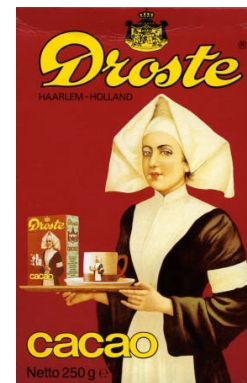
[usually we decrease N for all recursive calls]

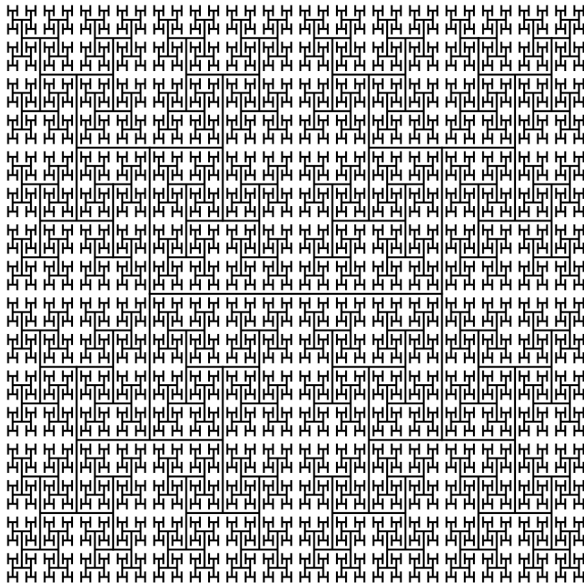
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## Recursive Graphics



New Yorker Magazine, August 11, 2008



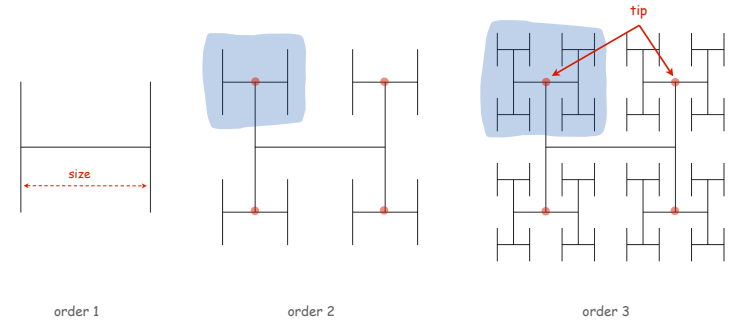


## Htree

### H-tree of order n.

- Draw an H.
- Recursively draw 4 H-trees of order n-1, one connected to each tip.

and half the size



## Htree in Java

```

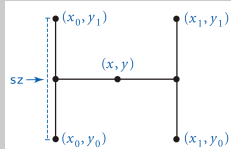
public class Htree
{
    public static void draw(int n, double sz, double x, double y)
    {
        if (n == 0) return;
        double x0 = x - sz/2, x1 = x + sz/2;
        double y0 = y - sz/2, y1 = y + sz/2;

        StdDraw.line(x0, y, x1, y);
        StdDraw.line(x0, y0, x0, y1); ← draw the H, centered on (x, y)
        StdDraw.line(x1, y0, x1, y1);

        draw(n-1, sz/2, x0, y0);
        draw(n-1, sz/2, x0, y1); ← recursively draw 4 half-size Hs
        draw(n-1, sz/2, x1, y0);
        draw(n-1, sz/2, x1, y1);
    }

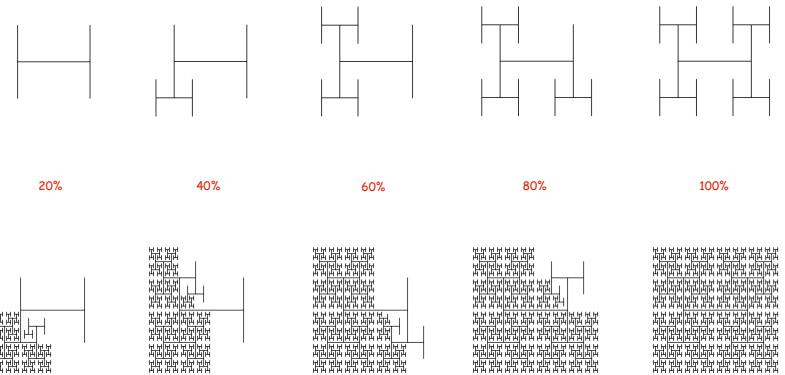
    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        draw(n, .5, .5, .5);
    }
}

```



## Animated H-tree

Animated H-tree. Pause for 1 second after drawing each H.



## Divide-and-Conquer

### Divide-and-conquer paradigm.

- Break up problem into smaller subproblems of same structure.
- Solve subproblems recursively using same method.
- Combine results to produce solution to original problem.

Divide et impera. Veni, vidi, vici. - Julius Caesar

### Many important problems succumb to divide-and-conquer.

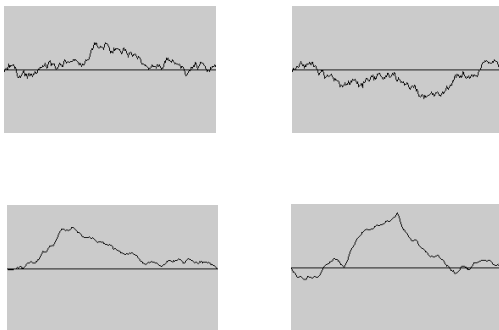
- FFT for signal processing.
- Parsers for programming languages.
- Multigrid methods for solving PDEs.
- **Quicksort and mergesort for sorting.**
- Hilbert curve for domain decomposition.
- Quad-tree for efficient N-body simulation.
- **Midpoint displacement method for fractional Brownian motion.**

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## Fractional Brownian Motion

### Physical process which models many natural and artificial phenomenon.

- Price of stocks.
- Dispersion of ink flowing in water.
- Rugged shapes of mountains and clouds.
- Fractal landscapes and textures for computer graphics.



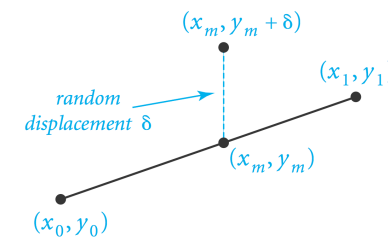
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## Application: Fractional Brownian Motion

### Simulating Brownian Motion

#### Midpoint displacement method.

- Maintain an interval with endpoints  $(x_0, y_0)$  and  $(x_1, y_1)$ .
- Divide the interval in half.
- Choose  $\delta$  at random from Gaussian distribution.
- Set  $x_m = (x_0 + x_1)/2$  and  $y_m = (y_0 + y_1)/2 + \delta$ .
- Recur on the left and right intervals.



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## Simulating Brownian Motion: Java Implementation

### Midpoint displacement method.

- Maintain an interval with endpoints  $(x_0, y_0)$  and  $(x_1, y_1)$ .
- Choose  $\delta$  at random from Gaussian distribution.
- Divide the interval in half: Set  $x_m = (x_0 + x_1)/2$  and  $y_m = (y_0 + y_1)/2 + \delta$ .
- Recur on the left and right intervals.

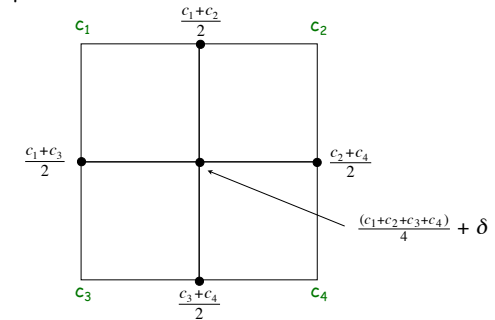
```
public static void curve(double x0, double y0,
                        double x1, double y1, double var)
{
    if (x1 - x0 < 0.01)
    {
        StdDraw.line(x0, y0, x1, y1);
        return;
    }
    double xm = (x0 + x1) / 2;
    double ym = (y0 + y1) / 2;
    ym += StdRandom.gaussian(0, Math.sqrt(var));
    curve(x0, y0, xm, ym, var/2);
    curve(xm, ym, x1, y1, var/2); ← variance halves at each level;
                                     change factor to get different shapes
}
```

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## Plasma Cloud

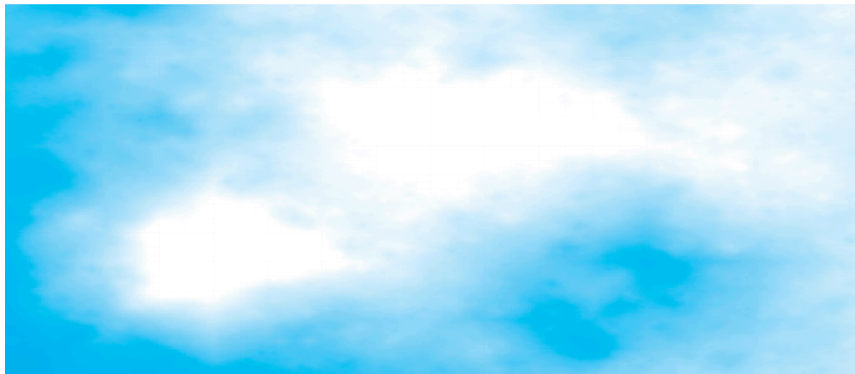
### Plasma cloud centered at $(x, y)$ of size $s$ .

- Each corner labeled with some grayscale value.
- Divide square into four quadrants.
- The grayscale of each new corner is the average of others.
  - center: average of the four corners + random displacement
  - others: average of two original corners
- Recur on the four quadrants.



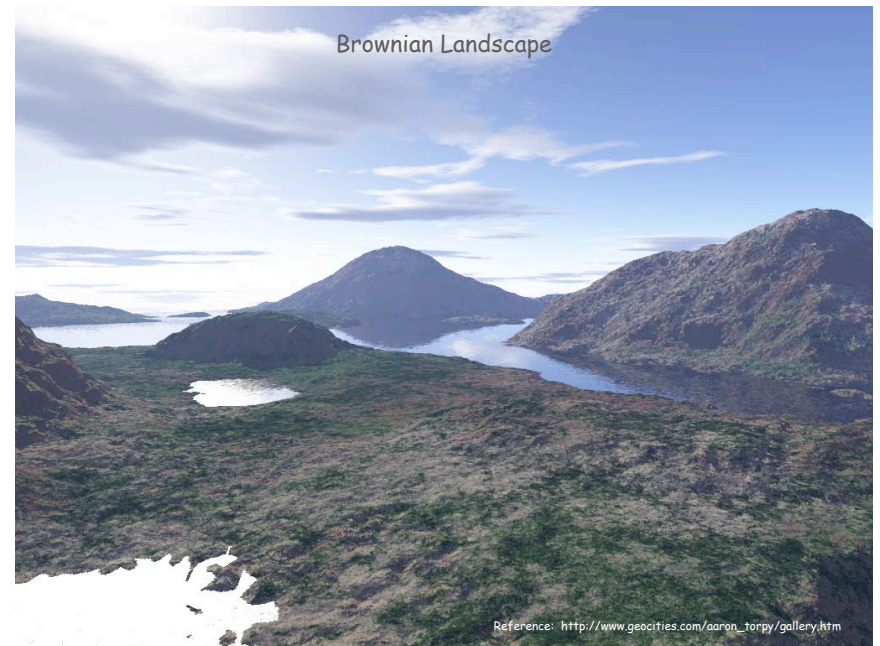
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## Plasma Cloud



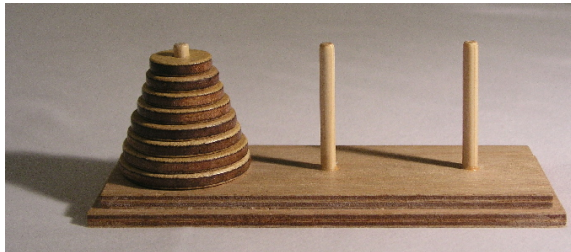
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## Brownian Landscape



Reference: [http://www.geocities.com/aaron\\_torpy/gallery.htm](http://www.geocities.com/aaron_torpy/gallery.htm)

## Towers of Hanoi



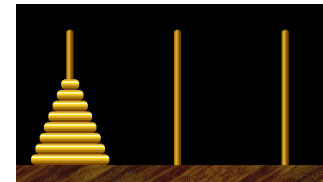
<http://en.wikipedia.org/wiki/Image:Hanoiklein.jpg>

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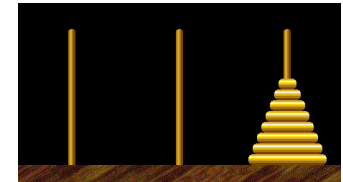
## Towers of Hanoi

Move all the discs from the leftmost peg to the rightmost one.

- Only one disc may be moved at a time.
- A disc can be placed either on empty peg or on top of a larger disc.



start



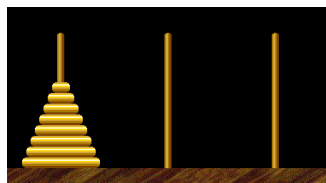
finish



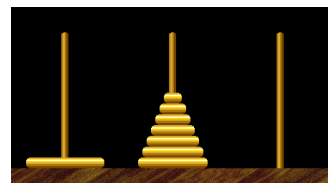
Edouard Lucas (1883)

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## Towers of Hanoi: Recursive Solution

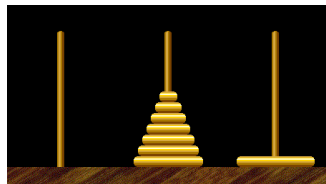


Move n-1 smallest discs right.

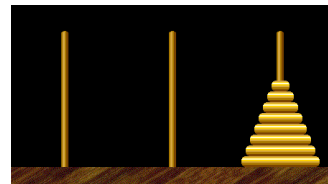


Move largest disc left.

cyclic wrap-around



Move n-1 smallest discs right.



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## Towers of Hanoi Legend

- Q. Is world going to end (according to legend)?
  - 64 golden discs on 3 diamond pegs.
  - World ends when certain group of monks accomplish task.
- Q. Will computer algorithms help?

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## Towers of Hanoi: Recursive Solution

```
public class TowersOfHanoi
{
    public static void moves(int n, boolean left)
    {
        if (n == 0) return;
        moves(n-1, !left);
        if (left) System.out.println(n + " left");
        else     System.out.println(n + " right");
        moves(n-1, !left);
    }

    public static void main(String[] args)
    {
        int N = Integer.parseInt(args[0]);
        moves(N, true);
    }
}
```

moves(n, true) : move discs 1 to n one pole to the left  
 moves(n, false): move discs 1 to n one pole to the right

← smallest disc

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## Towers of Hanoi: Recursive Solution

```
% java TowersOfHanoi 3
1 left
2 right
1 left
3 left
1 left
2 right
1 left

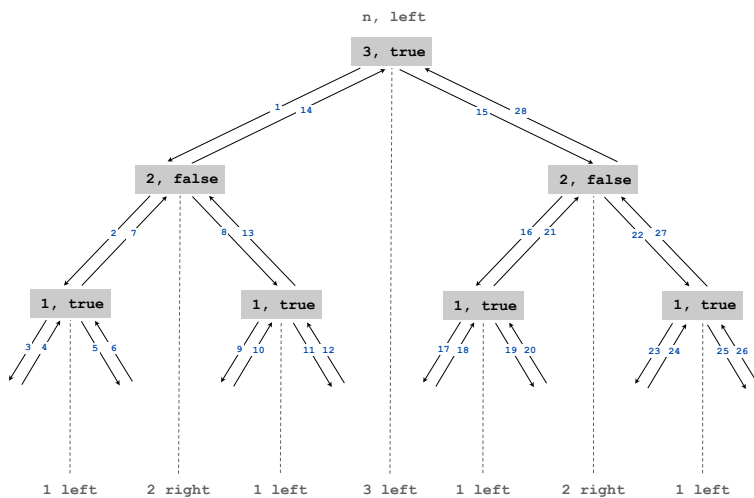
% java TowersOfHanoi 4
1 right
2 left
2 left
1 right
4 left
1 right
2 left
1 right
3 right
1 right
2 left
1 right
```

every other move is smallest disc

↑  
subdivisions  
of  
ruler

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## Towers of Hanoi: Recursion Tree



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## Towers of Hanoi: Properties of Solution

### Remarkable properties of recursive solution.

- Takes  $2^n - 1$  moves to solve n disc problem.
- Sequence of discs is same as subdivisions of ruler.
- Every other move involves smallest disc.

### Recursive algorithm yields non-recursive solution!

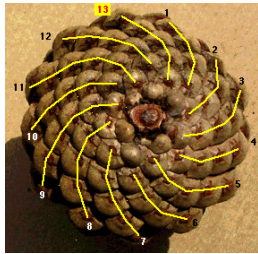
- Alternate between two moves:
    - move smallest disc to right if n is even
    - make only legal move not involving smallest disc
- ← to left if n is odd

### Recursive algorithm may reveal fate of world.

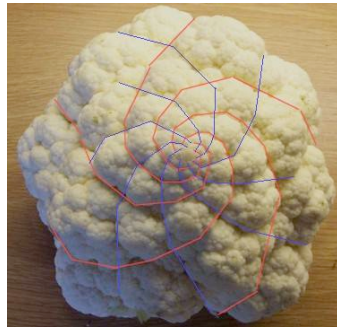
- Takes 585 billion years for n = 64 (at rate of 1 disc per second).
- Reassuring fact: any solution takes at least this long!

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## Fibonacci Numbers



pinecone



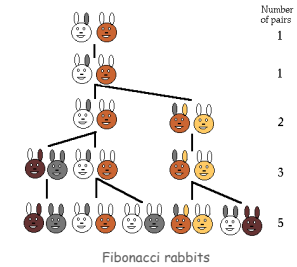
cauliflower

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## Fibonacci Numbers

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$



L. P. Fibonacci  
(1170 - 1250)

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## A Possible Pitfall With Recursion

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

A natural for recursion?

```
public static long F(int n)
{
    if (n == 0) return 0;
    if (n == 1) return 1;
    return F(n-1) + F(n-2);
}
```

FVI (classical math):

$$F(n) = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$$

$$= \lfloor \phi^n / \sqrt{5} \rfloor$$

$\phi$  = golden ratio = 1.618

Ex:  $F(50) \approx 1.2 \times 10^{10}$

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## TEQ on Recursion 1.1 (difficult but important)

Is this an efficient way to compute  $F(50)$ ?

```
public static long F(int n)
{
    if (n == 0) return 0;
    if (n == 1) return 1;
    return F(n-1) + F(n-2);
}
```

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## TEQ on Recursion 1.2 (easy and also important)

Is this an efficient way to compute  $F(50)$ ?

```
long[] F = new long[51];
F[0] = 0; F[1] = 1;
if (n == 1) return 1;
for (int i = 2; i <= 50; i++)
    F[i] = F[i-1] + F[i-2];
```

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## Summary

How to write simple recursive programs?

- Base case, reduction step.
- Trace the execution of a recursive program.
- Use pictures.

Why learn recursion?

- New mode of thinking.
- Powerful programming tool.

Towers of Hanoi by W. A. Schloss.

Divide-and-conquer. Elegant solution to many important problems.

Exponential time.

- Easy to specify recursive program that takes exponential time.
- Don't do it unless you plan to (and are working on a small problem).

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