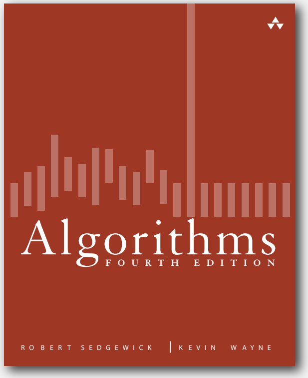
Geometric Applications of BSTs

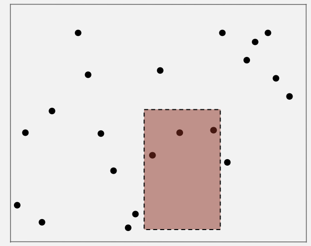


- 1d range search
- line segment intersection
- kd trees
- interval search trees
- rectangle intersection

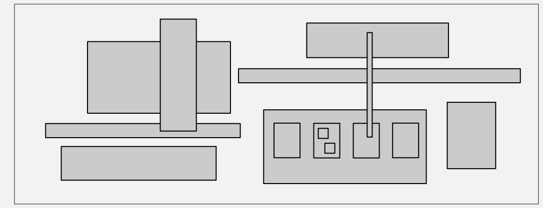
Overview

This lecture. Intersections among geometric objects.

Applications. CAD, games, movies, virtual reality, VLSI design, databases, ….



2d orthogonal range search



orthogonal rectangle intersection

Efficient solutions. Binary search trees (and extensions).

- 1d range search
- line segment intersection
- kd trees
- interval search trees
- rectangle intersection

1d range search

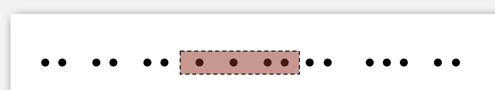
Extension of ordered symbol table.

- Insert key-value pair.
- Search for key k.
- Delete key k.
- Range search: find all keys between k_1 and k_2.
- Range count: number of keys between k_1 and k_2.

Application. Database queries.

Geometric interpretation.

- Keys are point on a line.
- Find/count points in a given 1d interval.



insert B	B
insert D	B D
insert A	A B D
insert I	A B D I
insert H	A B D H I
insert F	A B D F H I
insert P	A B D F H I P
count G to K	2
search G to K	H I

1d range search: implementations

Unordered array. Fast insert, slow range search.

Ordered array. Slow insert, binary search for k_1 and k_2 to do range search.

data structure	insert	range count	range search
unordered array	1	N	N
ordered array	N	log N	R + log N
goal	log N	log N	R + log N

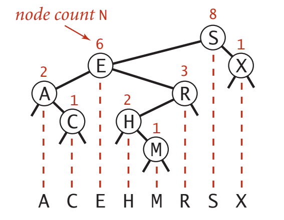
order of growth of running time for 1d range search

Parameters.

- N = number of keys.
- R = number of keys that match. ← running time is output sensitive (number of matching keys can be N)

1d range count: BST implementation

1d range count. How many keys between `lo` and `hi` ?




```
public int size(Key lo, Key hi)
{
   if (contains(hi)) return rank(hi) - rank(lo) + 1;
   else              return rank(hi) - rank(lo);
}
```

number of keys < hi

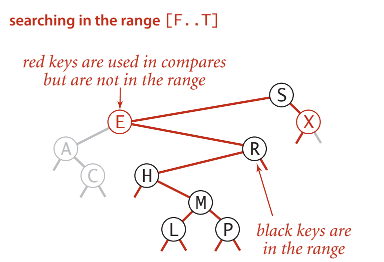
Proposition. Running time is proportional to $\log N$ (assuming BST is balanced).

Pf. Nodes examined = search path to `lo` + search path to `hi`.

1d range search: BST implementation

1d range search. Find all keys between `lo` and `hi`.

- Recursively find all keys in left subtree (if any could fall in range).
- Check key in current node.
- Recursively find all keys in right subtree (if any could fall in range).



Range search in a BST

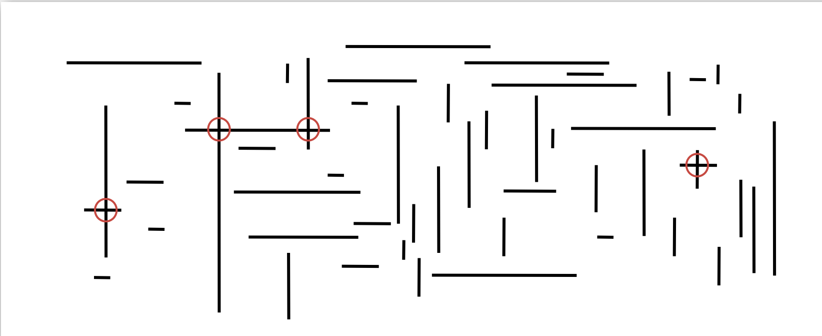
Proposition. Running time is proportional to $R + \log N$ (assuming BST is balanced).

Pf. Nodes examined = search path to `lo` + search path to `hi` + matching keys.

- 1d range search
- **line segment intersection**
- kd trees
- interval search trees
- rectangle intersection

Orthogonal line segment intersection search

Given N horizontal and vertical line segments, find all intersections.



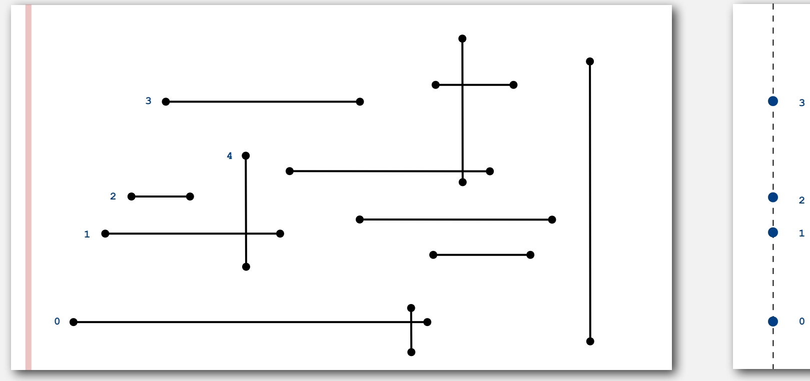
Nondegeneracy assumption. All x- and y-coordinates are distinct.

Quadratic algorithm. Check all pairs of line segments for intersection.

Orthogonal line segment intersection search: sweep-line algorithm

Sweep vertical line from left to right.

- x-coordinates define events.
- h-segment (left endpoint): insert y-coordinate into BST.

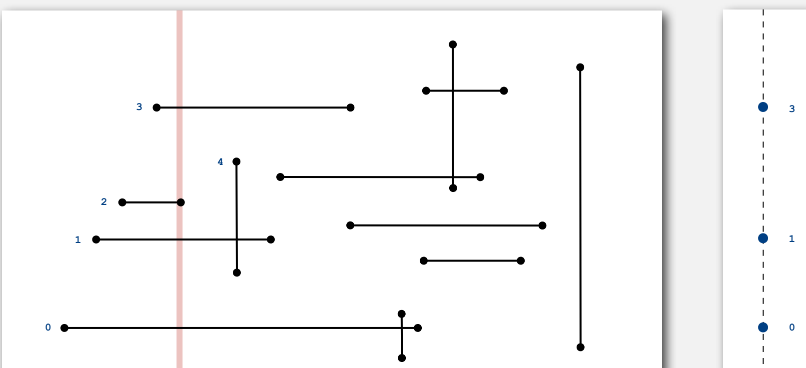


y−coordinates

Orthogonal line segment intersection search: sweep-line algorithm

Sweep vertical line from left to right.

- x-coordinates define events.
- h-segment (left endpoint): insert y-coordinate into BST.
- h-segment (right endpoint): remove y-coordinate from BST.

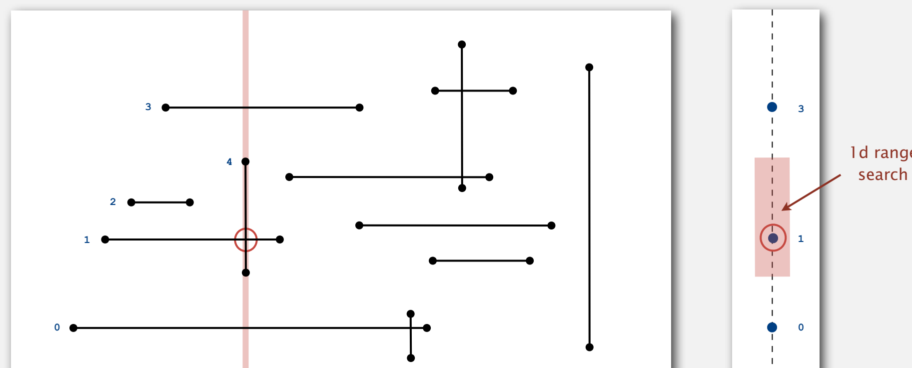


y−coordinates

Orthogonal line segment intersection search: sweep-line algorithm

Sweep vertical line from left to right.

- x-coordinates define events.
- h-segment (left endpoint): insert y-coordinate into BST.
- h-segment (right endpoint): remove y-coordinate from BST.
- v-segment: range search for interval of y-endpoints.



y−coordinates

Orthogonal line segment intersection search: sweep-line algorithm analysis

Proposition. The sweep-line algorithm takes time proportional to $N \log N + R$ to find all R intersections among N orthogonal line segments.

Pf.

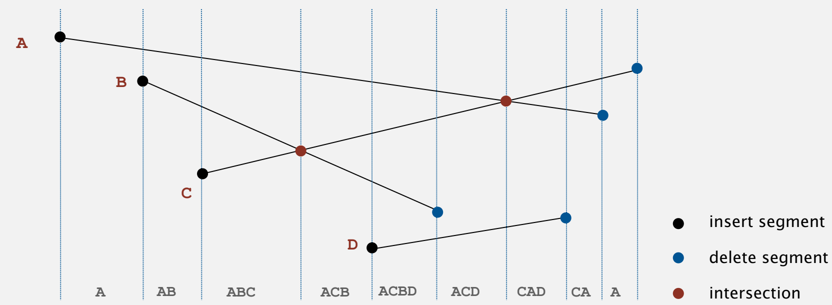
- Put x-coordinates on a PQ (or sort). ← N log N
- Insert y-coordinates into BST. ← N log N
- Delete y-coordinates from BST. ← N log N
- Range searches in BST. ← N log N + R

Bottom line. Sweep line reduces 2d orthogonal line segment intersection search to 1d range search.

General line segment intersection search

Sweep-line algorithm.

- Maintain segments that intersect sweep line ordered by y-coordinate.
- Intersections can only occur between adjacent segments.
- Delete/add line segment ⇒ one/two new pairs of adjacent segments.
- Intersection ⇒ swap adjacent segments.



order of segments that intersect sweep line

General line segment intersection search: implementation

Sweep-line algorithm.

- Maintain PQ of important x-coordinates: endpoints and intersections.
- Maintain set of segments intersecting sweep line, in BST sorted by y-coordinates.

to support "next largest" and "next smallest" queries

Proposition. The sweep-line algorithm takes time proportional to $R \log N + N \log N$ to find all R intersections among N orthogonal line segments.

Implementation issues.

- Degeneracy.
- Floating-point precision.
- Must use PQ, not presort (intersection events are unknown ahead of time).

- 1d range search
- line segment intersection
- **kd trees**
- interval search trees
- rectangle intersection

2-d orthogonal range search

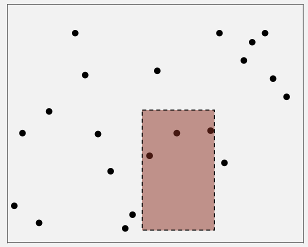
Extension of ordered symbol-table to 2d keys.

- Insert a 2d key.
- Delete a 2d key.
- Search for a 2d key.
- Range search: find all keys that lie in a 2d range.
- Range count: number of keys that lie in a 2d range.

Geometric interpretation.

- Keys are point in the plane.
- Find/count points in a given h–v rectangle.

rectangle is axis-aligned

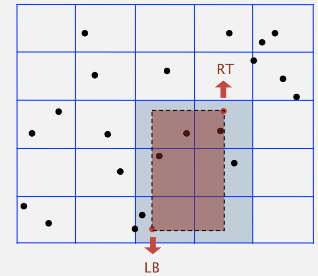


Applications. Networking, circuit design, databases.

2d orthogonal range search: grid implementation

Grid implementation.

- Divide space into M-by-M grid of squares.
- Create list of points contained in each square.
- Use 2d array to directly index relevant square.
- Insert: add (x, y) to list for corresponding square.
- Range search: examine only those squares that intersect 2d range query.



2d orthogonal range search: grid implementation costs

Space-time tradeoff.

- Space: $M^2 + N$.
- Time: $1 + N / M^2$ per square examined, on average.

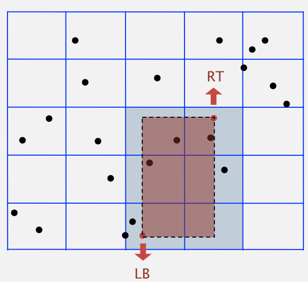
Choose grid square size to tune performance.

- Too small: wastes space.
- Too large: too many points per square.
- Rule of thumb: $\sqrt{N}$-by-$\sqrt{N}$ grid.

Running time. [if points are evenly distributed]

- Initialize data structure: N.
- Insert point: 1.
- Range search: 1 per point in range.

choose M ~ √N

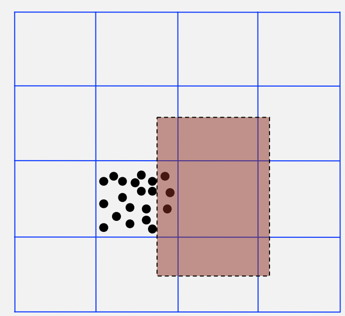


Clustering

Grid implementation. Fast and simple solution for evenly-distributed points.

Problem. Clustering a well-known phenomenon in geometric data.

- Lists are too long, even though average length is short.
- Need data structure that gracefully adapts to data.

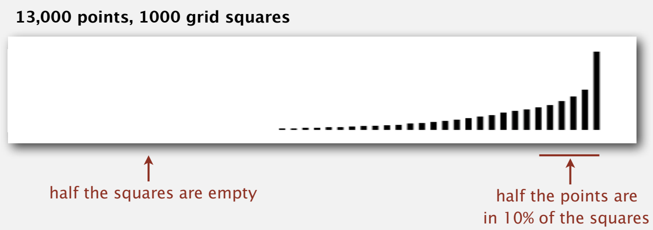


Clustering

Grid implementation. Fast and simple solution for evenly-distributed points.

Problem. Clustering a well-known phenomenon in geometric data.

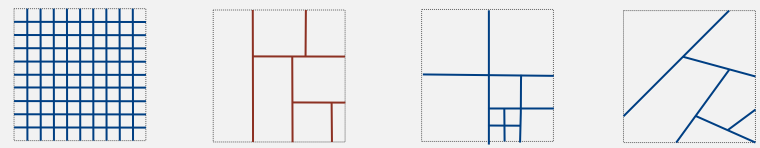
Ex. USA map data.



Space-partitioning trees

Use a tree to represent a recursive subdivision of 2d space.

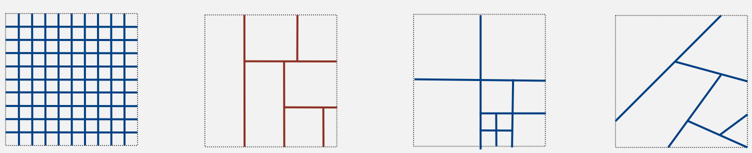
Grid. Divide space uniformly into squares.
2d tree. Recursively divide space into two halfplanes.
Quadtree. Recursively divide space into four quadrants.
BSP tree. Recursively divide space into two regions.



Grid | 2d tree | Quadtree | BSP tree

Space-partitioning trees: applications

Applications.
- Ray tracing.
- 2d range search.
- Flight simulators.
- N-body simulation.
- Collision detection.
- Astronomical databases.
- Nearest neighbor search.
- Adaptive mesh generation.
- Accelerate rendering in Doom.
- Hidden surface removal and shadow casting.

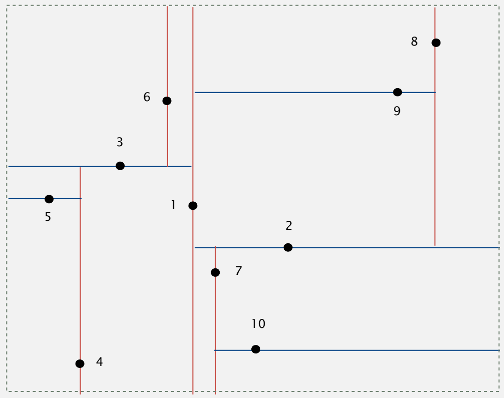


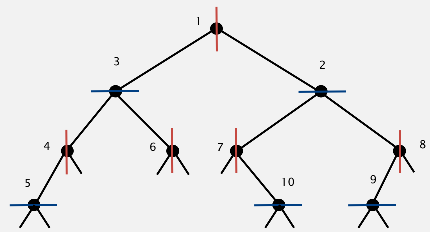
Grid | 2d tree | Quadtree | BSP tree

2d tree demo

2d tree

Recursively partition plane into two halfplanes.

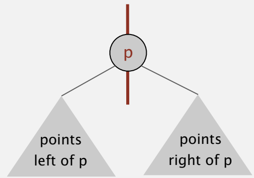


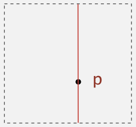


2d tree implementation

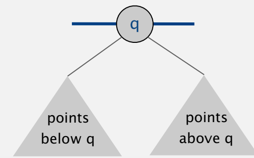
Data structure. BST, but alternate using x- and y-coordinates as key.

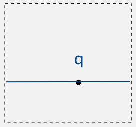
- Search gives rectangle containing point.
- Insert further subdivides the plane.



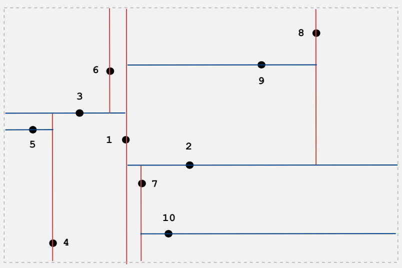


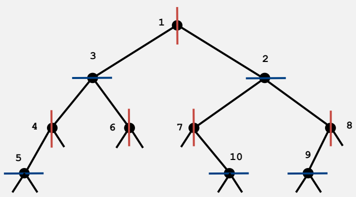
even levels





odd levels





2d tree demo: 2d orthogonal range search and nearest neighbor search

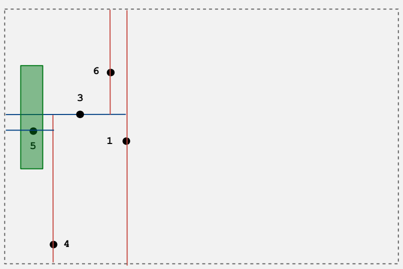
2d tree: 2d orthogonal range search

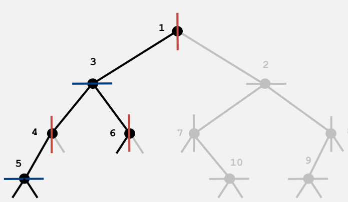
Range search. Find all points in a query axis-aligned rectangle.

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).

Typical case. $R + \log N$.

Worst case (assuming tree is balanced). $R + \sqrt{N}$.





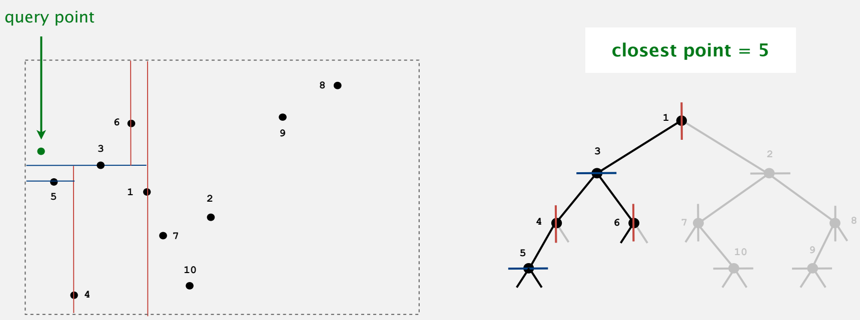
2d tree: nearest neighbor search

Nearest neighbor search. Given a query point, find the closest point.

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.

Typical case. $\log N$.

Worst case (even if tree is balanced). N.



Flocking birds

Q. What "natural algorithm" do starlings, migrating geese, starlings, cranes, bait balls of fish, and flashing fireflies use to flock?

http://www.youtube.com/watch?v=XH-groCeKbE

Flocking boids [Craig Reynolds, 1986]

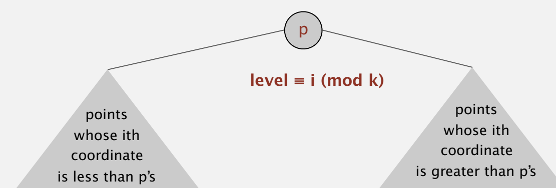
Boids. Three simple rules lead to complex emergent flocking behavior:

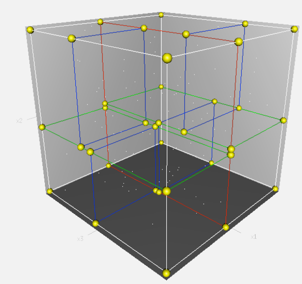
- Collision avoidance: point away from k nearest boids.
- Flock centering: point towards the center of mass of k nearest boids.
- Velocity matching: update velocity to the average of k nearest boids.

Kd tree

Kd tree. Recursively partition k-dimensional space into 2 halfspaces.

Implementation. BST, but cycle through dimensions ala 2d trees.





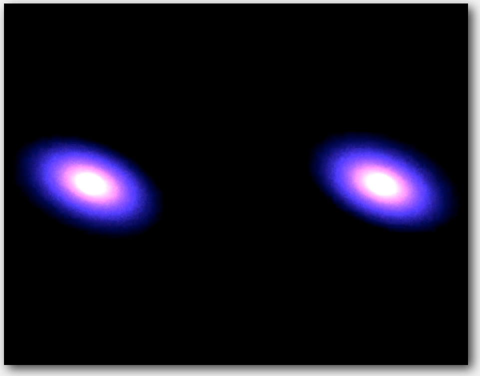
Efficient, simple data structure for processing k-dimensional data.

- Widely used.
- Adapts well to high-dimensional and clustered data.
- Discovered by an undergrad in an algorithms class!

Jon Bentley

N-body simulation

Goal. Simulate the motion of N particles, mutually affected by gravity.



http://www.youtube.com/watch?v=ua7YlN4eL_w

Brute force. For each pair of particles, compute force. $F = \frac{G\, m_1\, m_2}{r^2}$

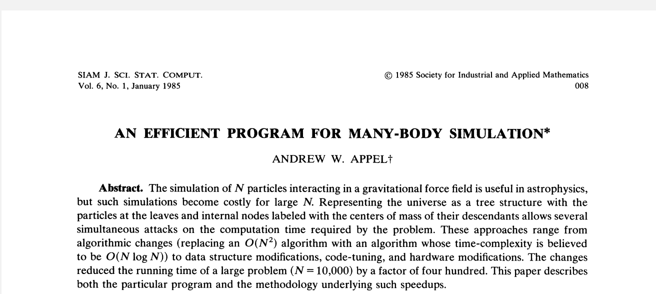
Appel algorithm for N-body simulation

Key idea. Suppose particle is far, far away from cluster of particles.

- Treat cluster of particles as a single aggregate particle.
- Compute force between particle and center of mass of aggregate particle.

Appel algorithm for N-body simulation

- Build 3d-tree with N particles as nodes.
- Store center-of-mass of subtree in each node.
- To compute total force acting on a particle, traverse tree, but stop as soon as distance from particle to subdivision is sufficiently large.



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AN EFFICIENT PROGRAM FOR MANY-BODY SIMULATION*

ANDREW W. APPEL†

Abstract. The simulation of N particles interacting in a gravitational force field is useful in astrophysics, but such simulations become costly for large N. Representing the universe as a tree structure with the particles at the leaves and internal nodes labeled with the centers of mass of their descendants allows several simultaneous attacks on the computation time required by the problem. These approaches range from algorithmic changes (replacing an $O(N^2)$ algorithm with an algorithm whose time-complexity is believed to be $O(N \log N)$) to data structure modifications, code-tuning, and hardware modifications. The changes reduced the running time of a large problem ($N = 10{,}000$) by a factor of four hundred. This paper describes both the particular program and the methodology underlying such speedups.

Impact. Running time per step is $N \log N$ instead of $N^2 \Rightarrow$ enables new research.

- 1d range search
- line segment intersection
- kd trees
- **interval search trees**
- rectangle intersection

1d interval search

1d interval search. Data structure to hold set of (overlapping) intervals.

- Insert an interval (*lo*, *hi*).
- Search for an interval (*lo*, *hi*).
- Delete an interval (*lo*, *hi*).
- Interval intersection query: given an interval (*lo*, *hi*), find all intervals in data structure overlapping (*lo*, *hi*).

Interval search trees

`public class IntervalST<Key extends Comparable<Key>, Value>`

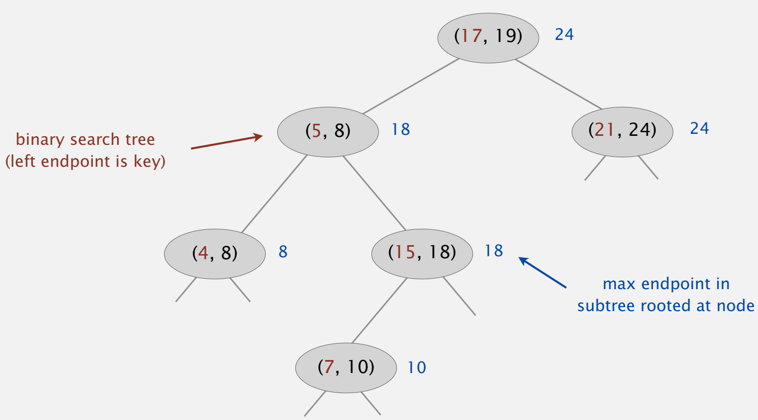
	`IntervalST()`	*create interval search tree*
`void`	`put(Key lo, Key hi, Value val)`	*put interval-value pair into ST*
`Value`	`get(Key lo, Key hi)`	*value paired with given interval*
`void`	`delete(Key lo, Key hi)`	*delete the given interval*
`Iterable<Value>`	`intersects(Key lo, Key hi)`	*all intervals that intersect the given interval*

Nondegeneracy assumption. No two intervals have the same left endpoint.

Interval search trees

Create BST, where each node stores an interval (*lo*, *hi*).

- Use left endpoint as BST key.
- Store max endpoint in subtree rooted at node.

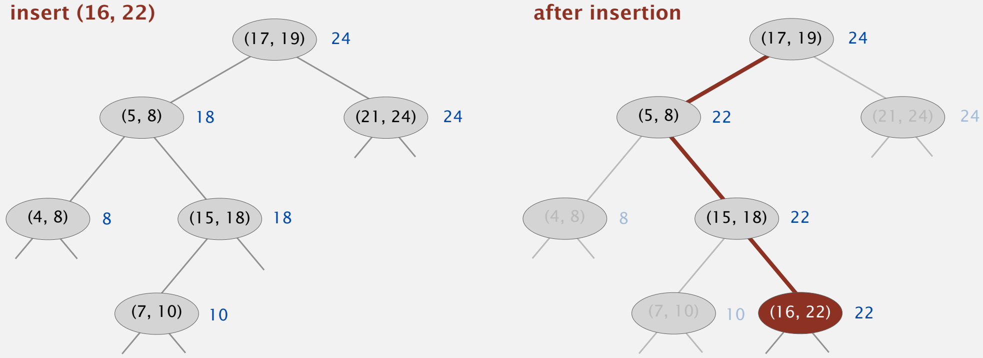


Interval search tree demo

Insert an interval

To insert an interval (lo, hi) :

- Insert into BST, using lo as the key.
- Update max in each node on search path.



Search for an intersecting interval

To search for an interval that intersects query interval (lo, hi) :

- Start at root.
- If subtree is empty, return not found.
- Else if interval in node intersects query interval, return it.
- Else if left subtree is empty, go right.
- Else if max endpoint in left subtree is less than lo, go right.
- Else go left.

```
Node x = root;
while (x != null)
{
   if      (x.interval.intersects(lo, hi)) return x.interval;
   else if (x.left == null)                x = x.right;
   else if (x.left.max < lo)               x = x.right;
   else                                    x = x.left;
}
return null;
```


Search for an intersecting interval

To search for an interval that intersects query interval (lo, hi) :

- If interval in node intersects query interval, return it.
- If left subtree is null, go right.
- If max endpoint in left subtree is less than lo, go right.
- Else go left.

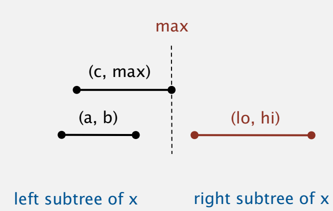
Case 1. If search goes right, then no intersection in left.

Pf.

- Left subtree is null $\Rightarrow$ trivial.
- Max endpoint max in left subtree is less than lo $\Rightarrow$ for any interval (a, b) in left subtree of x, we have $b \leq max < lo$.

definition of max

reason for going right



Search for an intersecting interval

To search for an interval that intersects query interval (lo, hi) :

- If interval in node intersects query interval, return it.
- If left subtree is null, go right.
- If max endpoint in left subtree is less than lo, go right.
- Else go left.

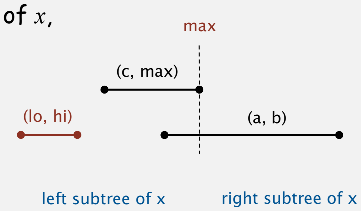
Case 2. If search goes left, then there is either an intersection in left subtree or no intersections in either.

Pf. Suppose no intersection in left.

- Since went left, we have $lo \leq max$.
- Then for any interval (a, b) in right subtree of x, $hi < c \leq a \Rightarrow$ no intersection in right.

no intersections in left subtree

intervals sorted by left endpoint



Interval search tree: analysis

Implementation. Use a red-black BST to guarantee performance.

can maintain auxiliary information using log N extra work per op

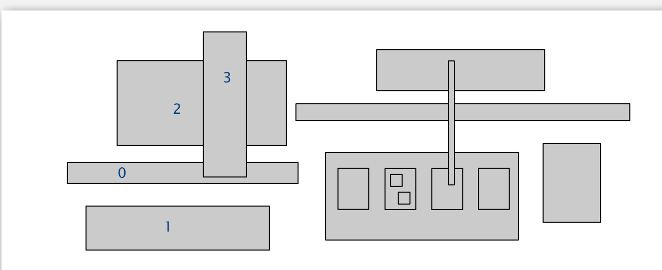
operation	brute	interval search tree	best in theory
insert interval	1	log N	log N
find interval	N	log N	log N
delete interval	N	log N	log N
find any interval that intersects (lo, hi)	N	log N	log N
find all intervals that intersects (lo, hi)	N	R log N	R + log N

order of growth of running time for N intervals

- 1d range search
- line segment intersection
- kd trees
- interval search trees
- **rectangle intersection**

Orthogonal rectangle intersection search

Goal. Find all intersections among a set of N orthogonal rectangles.



Non-degeneracy assumption. All x- and y-coordinates are distinct.

Quadratic algorithm. Check all pairs of rectangles for intersection.

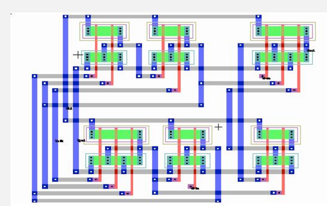
Microprocessors and geometry

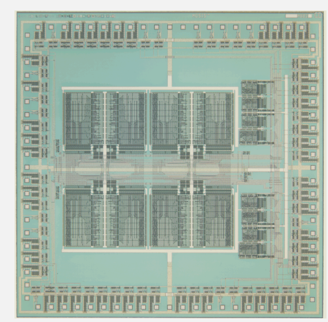
Early 1970s. microprocessor design became a geometric problem.

- Very Large Scale Integration (VLSI).
- Computer-Aided Design (CAD).

Design-rule checking.

- Certain wires cannot intersect.
- Certain spacing needed between different types of wires.
- Debugging = orthogonal rectangle intersection search.

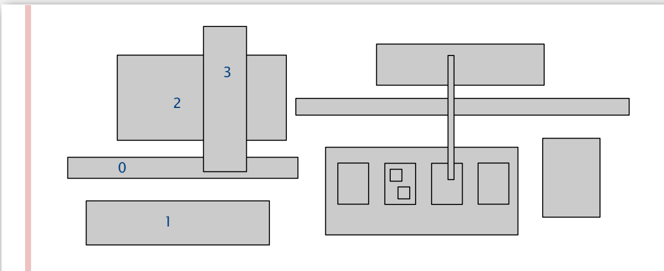


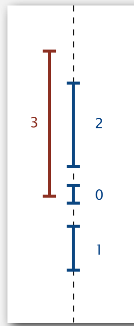


Orthogonal rectangle intersection search: sweep-line algorithm

Sweep vertical line from left to right.

- x-coordinates of left and right endpoints define events.
- Maintain set of rectangles that intersect the sweep line in an interval search tree (using y-intervals of rectangle).
- Left endpoint: interval search for y-interval of rectangle; insert y-interval.
- Right endpoint: remove y-interval.





y-coordinates

Orthogonal rectangle intersection search: sweep-line algorithm analysis

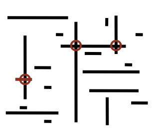
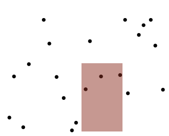
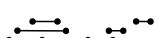
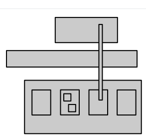
Proposition. Sweep line algorithm takes time proportional to $N \log N + R \log N$ to find R intersections among a set of N rectangles.

Pf.

- Put x-coordinates on a PQ (or sort). ⟵ N log N
- Insert y-intervals into ST. ⟵ N log N
- Delete y-intervals from ST. ⟵ N log N
- Interval searches for y-intervals. ⟵ N log N + R log N

Bottom line. Sweep line reduces 2d orthogonal rectangle intersection search to 1d interval search.

Geometric applications of BSTs

problem	example	solution
1d range search		BST
2d orthogonal line segment intersection search		sweep line reduces to 1d range search
kd range search		kd tree
1d interval search		interval search tree
2d orthogonal rectangle intersection search		sweep line reduces to 1d interval search