

1.3 BAGS, QUEUES, AND STACKS



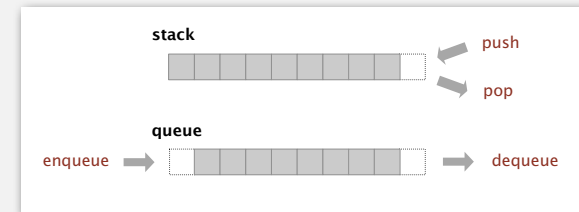
- ▶ stacks
- ▶ resizing arrays
- ▶ queues
- ▶ generics
- ▶ iterators
- ▶ applications

Algorithms, 4th Edition · Robert Sedgwick and Kevin Wayne · Copyright © 2002–2011 · September 27, 2011 3:35:54 PM

Stacks and queues

Fundamental data types.

- Value: collection of objects.
- Operations: **insert**, **remove**, **iterate**, test if empty.
- Intent is clear when we insert.
- Which item do we remove?



Stack. Examine the item most recently added. ← LIFO = "last in first out"

Queue. Examine the item least recently added. ← FIFO = "first in first out"

Client, implementation, interface

Separate interface and implementation.

Ex: stack, queue, bag, priority queue, symbol table, union-find,

Benefits.

- Client can't know details of implementation ⇒ client has many implementation from which to choose.
- Implementation can't know details of client needs ⇒ many clients can re-use the same implementation.
- **Design:** creates modular, reusable libraries.
- **Performance:** use optimized implementation where it matters.

Client: program using operations defined in interface.

Implementation: actual code implementing operations.

Interface: description of data type, basic operations.

- ▶ stacks
- ▶ resizing arrays
- ▶ queues
- ▶ generics
- ▶ iterators
- ▶ applications

Warmup API. Stack of strings data type.

```
public class StackOfStrings
```

```
    StackOfStrings()    create an empty stack
```

```
    void push(String s) insert a new item onto stack
```

```
    String pop()        remove and return the item  
                        most recently added
```

```
    boolean isEmpty()   is the stack empty?
```

```
    int size()          number of items on the stack
```

push pop



Warmup client. Reverse sequence of strings from standard input.

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```
public static void main(String[] args)
{
    StackOfStrings stack = new StackOfStrings();
    while (!StdIn.isEmpty())
    {
        String item = StdIn.readString();
        if (item.equals("-")) StdOut.print(stack.pop());
        else stack.push(item);
    }
}
```

push pop



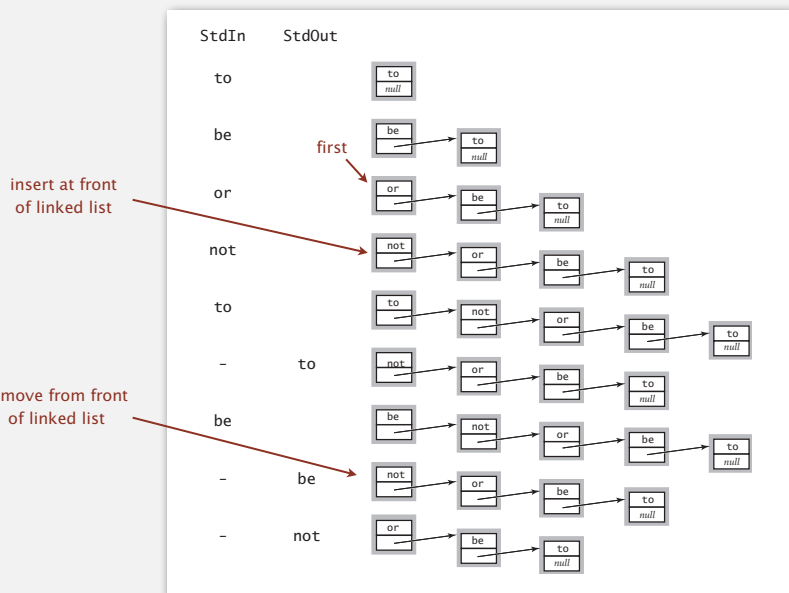
```
% more tobe.txt
to be or not to - be - - that - - - is

% java StackOfStrings < tobe.txt
to be not that or be
```

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Stack: linked-list representation

Maintain pointer to first node in a linked list; insert/remove from front.



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Stack pop: linked-list implementation

```
save item to return
String item = first.item;
```

```
delete first node

first = first.next;
```

inner class

```
public class Node
{
    String item;
    Node next;
    ...
}
```

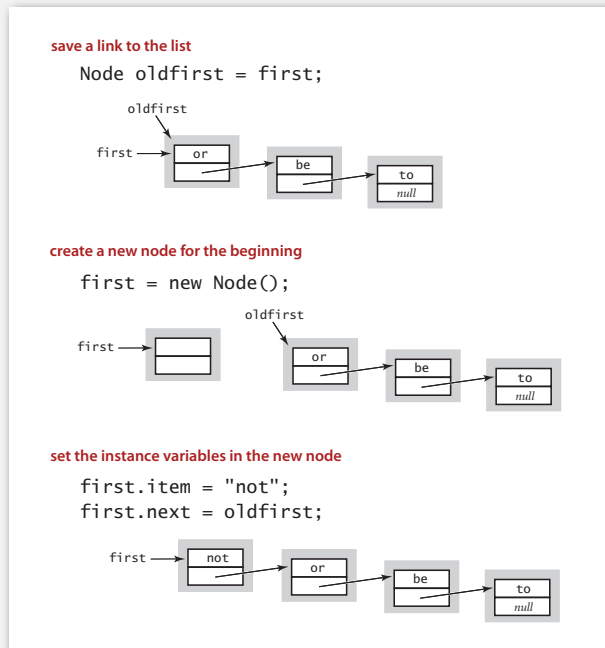
```
return saved item
return item;
```

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Stack push: linked-list implementation

inner class

```
public class Node
{
    String item;
    Node next;
    ...
}
```



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Stack: linked-list implementation in Java

```
public class LinkedStackOfStrings
{
    private Node first = null;

    private class Node
    {
        String item;
        Node next;
    }

    public boolean isEmpty()
    { return first == null; }

    public void push(String item)
    {
        Node oldfirst = first;
        first = new Node();
        first.item = item;
        first.next = oldfirst;
    }

    public String pop()
    {
        String item = first.item;
        first = first.next;
        return item;
    }
}
```

inner class

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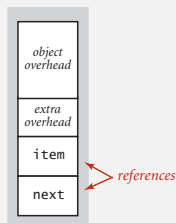
Stack: linked-list implementation performance

Proposition. Every operation takes constant time in the worst case.

Proposition. A stack with N items uses $\sim 40N$ bytes.

inner class

```
public class Node
{
    String item;
    Node next;
    ...
}
```



16 bytes (object overhead)

8 bytes (inner class extra overhead)

8 bytes (reference to String)

8 bytes (reference to Node)

40 bytes per stack node

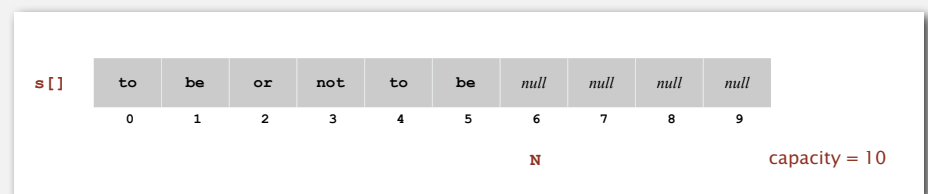
Remark. Analysis includes memory for the stack (but not the strings themselves, which the client owns).

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Stack: array implementation

Array implementation of a stack.

- Use array $s[]$ to store N items on stack.
- `push()`: add new item at $s[N]$.
- `pop()`: remove item from $s[N-1]$.



Defect. Stack overflows when N exceeds capacity. [stay tuned]

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Stack: array implementation

```
public class FixedCapacityStackOfStrings
{
    private String[] s;
    private int N = 0;

    public FixedCapacityStackOfStrings(int capacity)
    { s = new String[capacity]; }

    public boolean isEmpty()
    { return N == 0; }

    public void push(String item)
    { s[N++] = item; }

    public String pop()
    { return s[--N]; }
}
```

use to index into array;
then increment N

decrement N;
then use to index into array

a cheat (stay tuned)

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Stack considerations

Overflow and underflow.

- Underflow: throw exception if pop from an empty stack.
- Overflow: use resizing array for array implementation. [stay tuned]

Loitering. Holding a reference to an object when it is no longer needed.

```
public String pop()
{ return s[--N]; }
```

loitering

```
public String pop()
{
    String item = s[--N];
    s[N] = null;
    return item;
}
```

this version avoids "loitering":
garbage collector can reclaim memory
only if no outstanding references

Null items. We allow null items to be inserted.

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- › stacks
- › resizing arrays
- › queues
- › generics
- › iterators
- › applications

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Stack: resizing-array implementation

Problem. Requiring client to provide capacity does not implement API!

Q. How to grow and shrink array?

First try.

- `push()`: increase size of array `s[]` by 1.
- `pop()`: decrease size of array `s[]` by 1.

Too expensive.

- Need to copy all item to a new array.
- Inserting first N items takes time proportional to $1 + 2 + \dots + N \sim N^2/2$.

↑
infeasible for large N

Challenge. Ensure that array resizing happens infrequently.

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Stack: resizing-array implementation

Q. How to grow array?

A. If array is full, create a new array of **twice** the size, and copy items.

"repeated doubling"

```
public ResizingArrayStackOfStrings()
{ s = new String[1]; }

public void push(String item)
{
    if (N == s.length) resize(2 * s.length);
    s[N++] = item;
}

private void resize(int capacity)
{
    String[] copy = new String[capacity];
    for (int i = 0; i < N; i++)
        copy[i] = s[i];
    s = copy;
}
```

cost of array resizing is now
 $2 + 4 + 8 + \dots + N \sim 2N$

Consequence. Inserting first N items takes time proportional to N (not N^2).

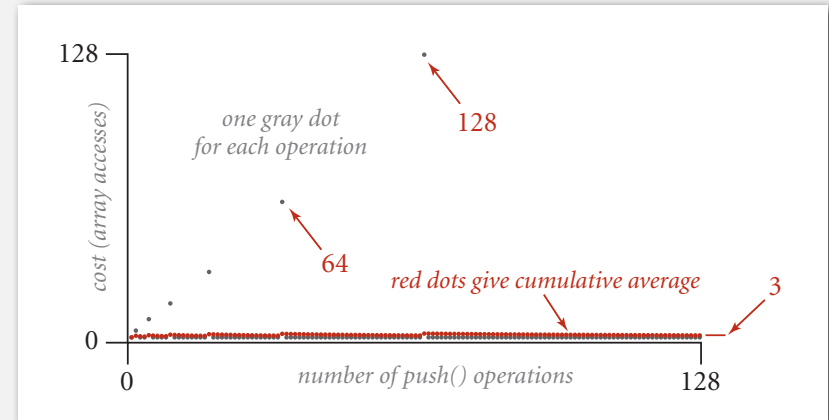
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Stack: amortized cost of adding to a stack

Cost of inserting first N items. $N + (2 + 4 + 8 + \dots + N) \sim 3N$.

1 array accesses
per push

k array accesses
to double to size k
(ignoring cost to create new array)



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Stack: resizing-array implementation

Q. How to shrink array?

First try.

- `push()`: double size of array `s[]` when array is full.
- `pop()`: halve size of array `s[]` when array is **one-half full**.

Too expensive in worst case.

"thrashing"

- Consider push-pop-push-pop... sequence when array is full.
- Each operation takes time proportional to N .

N = 5	to	be	or	not	to	null	null	null
N = 4	to	be	or	not				
N = 5	to	be	or	not	to	null	null	null
N = 4	to	be	or	not				

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Stack: resizing-array implementation

Q. How to shrink array?

Efficient solution.

- `push()`: double size of array `s[]` when array is full.
- `pop()`: halve size of array `s[]` when array is **one-quarter full**.

```
public String pop()
{
    String item = s[--N];
    s[N] = null;
    if (N > 0 && N == s.length/4) resize(s.length/2);
    return item;
}
```

Invariant. Array is between 25% and 100% full.

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Stack: resizing-array implementation trace

push()	pop()	N	a.length	a[]							
				0	1	2	3	4	5	6	7
		0	1	null							
to		1	1	to							
be		2	2	to be							
or		3	4	to be or							
not		4	4	to be or not							
to		5	8	to be or not	to	null	null	null			
-	to	4	8	to be or not	null	null	null	null			
be		5	8	to be or not	be	null	null	null			
-	be	4	8	to be or not	null	null	null	null			
-	not	3	8	to be or	null	null	null	null			
that		4	8	to be or	that	null	null	null			
-	that	3	8	to be or	null	null	null	null			
-	or	2	4	to be	null	null					
-	be	1	2	to	null						
is		2	2	to	is						

Trace of array resizing during a sequence of push() and pop() operations

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Stack resizing-array implementation: performance

Amortized analysis. Average running time per operation over a worst-case sequence of operations.

Proposition. Starting from an empty stack, any sequence of M push and pop operations takes time proportional to M .

	best	worst	amortized
construct	1	1	1
push	1	N	1
pop	1	N	1
size	1	1	1

order of growth of running time for resizing stack with N items

doubling and halving operations

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Stack resizing-array implementation: memory usage

Proposition. Uses between $\sim 8N$ and $\sim 32N$ bytes to represent a stack with N items.

- $\sim 8N$ when full.
- $\sim 32N$ when one-quarter full.

```
public class ResizingArrayStackOfStrings
{
    private String[] s;
    private int N = 0;
    ...
}
```

8 bytes (reference to array)
 24 bytes (array overhead)
 8 bytes \times array size
 4 bytes (int)
 4 bytes (padding)

Remark. Analysis includes memory for the stack (but not the strings themselves, which the client owns).

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Stack implementations: resizing array vs. linked list

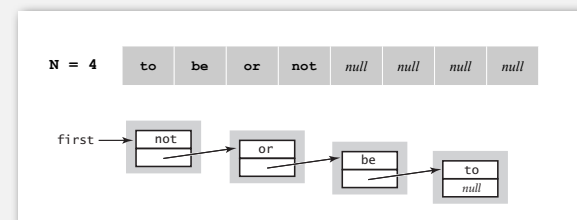
Tradeoffs. Can implement a stack with either a resizing array or a linked list; client can use interchangeably. Which one is better?

Linked-list implementation.

- Every operation takes constant time in the **worst case**.
- Uses extra time and space to deal with the links.

Resizing-array implementation.

- Every operation takes constant **amortized time**.
- Less wasted space.

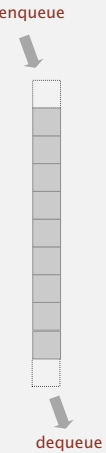


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- › stacks
- › resizing arrays
- › **queues**
- › generics
- › iterators
- › applications

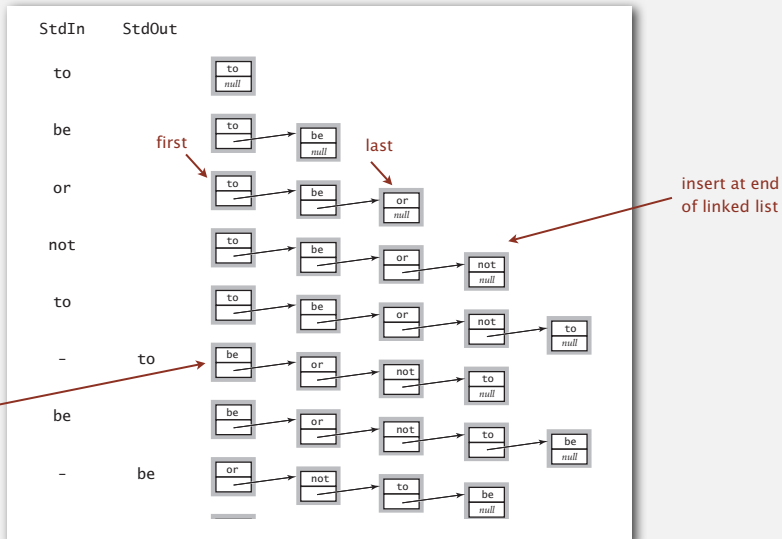
Queue API

```
public class QueueOfStrings
{
    QueueOfStrings ()           create an empty queue
    void enqueue (String s)     insert a new item onto queue
    String dequeue ()           remove and return the item
                                least recently added
    boolean isEmpty ()          is the queue empty?
    int size ()                 number of items on the queue
}
```



Queue: linked-list representation

Maintain pointer to first and last nodes in a linked list;
insert/remove from opposite ends.



Queue dequeue: linked-list implementation

```
save item to return
String item = first.item;

delete first node
first = first.next;

inner class
public class Node
{
    String item;
    Node next;
    ...
}

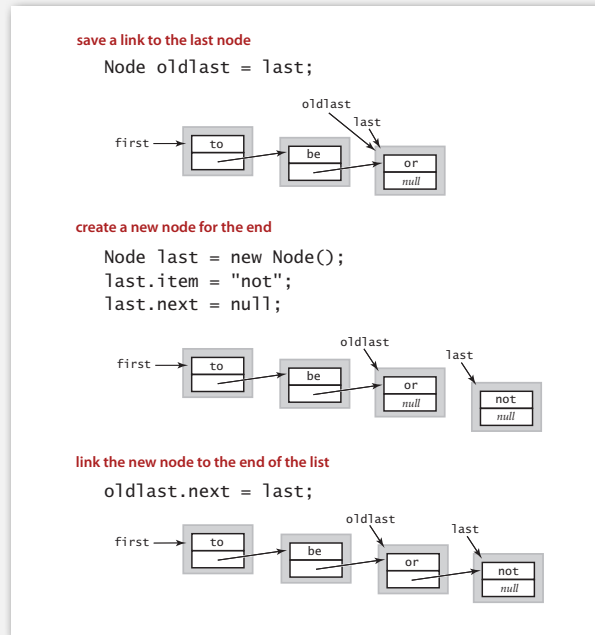
return saved item
return item;
```

Remark. Identical code to linked-list stack pop().

Queue enqueue: linked-list implementation

inner class

```
public class Node
{
    String item;
    Node next;
    ...
}
```



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Queue: linked-list implementation in Java

```
public class LinkedQueueOfStrings
{
    private Node first, last;

    private class Node
    { /* same as in StackOfStrings */ }

    public boolean isEmpty()
    { return first == null; }

    public void enqueue(String item)
    {
        Node oldlast = last;
        last = new Node();
        last.item = item;
        last.next = null;
        if (isEmpty()) first = last;
        else oldlast.next = last;
    }

    public String dequeue()
    {
        String item = first.item;
        first = first.next;
        if (isEmpty()) last = null;
        return item;
    }
}
```

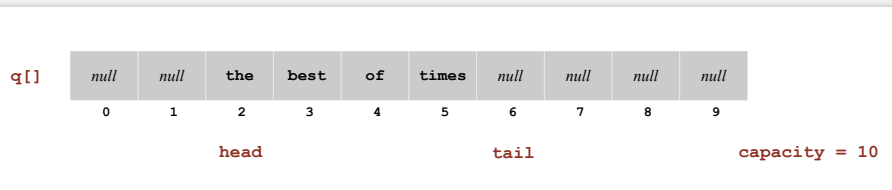
special cases for empty queue

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Queue: resizing array implementation

Array implementation of a queue.

- Use array `q[]` to store items in queue.
- `enqueue()`: add new item at `q[tail]`.
- `dequeue()`: remove item from `q[head]`.
- Update `head` and `tail` modulo the capacity.
- Add resizing array.



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- stacks
- resizing arrays
- queues
- **generics**
- iterators
- applications

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Parameterized stack

We implemented: `StackOfStrings`.

We also want: `StackOfURLs`, `StackOfInts`, `StackOfVans`,

Attempt 1. Implement a separate stack class for each type.

- Rewriting code is tedious and error-prone.
- Maintaining cut-and-pasted code is tedious and error-prone.

@#\$\$*! most reasonable approach until Java 1.5.



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Parameterized stack

We implemented: `StackOfStrings`.

We also want: `StackOfURLs`, `StackOfInts`, `StackOfVans`,

Attempt 2. Implement a stack with items of type `Object`.

- Casting is required in client.
- Casting is error-prone: run-time error if types mismatch.

```
StackOfObjects s = new StackOfObjects();
Apple a = new Apple();
Orange b = new Orange();
s.push(a);
s.push(b);
a = (Apple) (s.pop());
```

run-time error



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Parameterized stack

We implemented: `StackOfStrings`.

We also want: `StackOfURLs`, `StackOfInts`, `StackOfVans`,

Attempt 3. Java generics.

- Avoid casting in client.
- Discover type mismatch errors at compile-time instead of run-time.

```
Stack<Apple> s = new Stack<Apple>();
Apple a = new Apple();
Orange b = new Orange();
s.push(a);
s.push(b);
a = s.pop();
```

type parameter

compile-time error

Guiding principles. Welcome compile-time errors; avoid run-time errors.

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Generic stack: linked-list implementation

```
public class LinkedStackOfStrings
{
    private Node first = null;

    private class Node
    {
        String item;
        Node next;
    }

    public boolean isEmpty()
    { return first == null; }

    public void push(String item)
    {
        Node oldfirst = first;
        first = new Node();
        first.item = item;
        first.next = oldfirst;
    }

    public String pop()
    {
        String item = first.item;
        first = first.next;
        return item;
    }
}
```

```
public class Stack<Item>
{
    private Node first = null;

    private class Node
    {
        Item item;
        Node next;
    }

    public boolean isEmpty()
    { return first == null; }

    public void push(Item item)
    {
        Node oldfirst = first;
        first = new Node();
        first.item = item;
        first.next = oldfirst;
    }

    public Item pop()
    {
        Item item = first.item;
        first = first.next;
        return item;
    }
}
```

generic type name

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Generic stack: array implementation

```
public class FixedCapacityStackOfStrings
{
    private String[] s;
    private int N = 0;

    public ..StackOfStrings(int capacity)
    { s = new String[capacity]; }

    public boolean isEmpty()
    { return N == 0; }

    public void push(String item)
    { s[N++] = item; }

    public String pop()
    { return s[--N]; }
}
```

the way it should be

```
public class FixedCapacityStack<Item>
{
    private Item[] s;
    private int N = 0;

    public FixedCapacityStack(int capacity)
    { s = new Item[capacity]; }

    public boolean isEmpty()
    { return N == 0; }

    public void push(Item item)
    { s[N++] = item; }

    public Item pop()
    { return s[--N]; }
}
```

@#\$%! generic array creation not allowed in Java

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Generic stack: array implementation

```
public class FixedCapacityStackOfStrings
{
    private String[] s;
    private int N = 0;

    public ..StackOfStrings(int capacity)
    { s = new String[capacity]; }

    public boolean isEmpty()
    { return N == 0; }

    public void push(String item)
    { s[N++] = item; }

    public String pop()
    { return s[--N]; }
}
```

the way it is

```
public class FixedCapacityStack<Item>
{
    private Item[] s;
    private int N = 0;

    public FixedCapacityStack(int capacity)
    { s = (Item[]) new Object[capacity]; }

    public boolean isEmpty()
    { return N == 0; }

    public void push(Item item)
    { s[N++] = item; }

    public Item pop()
    { return s[--N]; }
}
```

the ugly cast

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Generic data types: autoboxing

Q. What to do about primitive types?

Wrapper type.

- Each primitive type has a **wrapper** object type.
- Ex: `Integer` is wrapper type for `int`.

Autoboxing. Automatic cast between a primitive type and its wrapper.

Syntactic sugar. Behind-the-scenes casting.

```
Stack<Integer> s = new Stack<Integer>();
s.push(17);      // s.push(new Integer(17));
int a = s.pop(); // int a = s.pop().intValue();
```

Bottom line. Client code can use generic stack for **any** type of data.

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- stacks
- resizing arrays
- queues
- generics
- iterators
- applications

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Iteration

Design challenge. Support iteration over stack items by client, without revealing the internal representation of the stack.



Java solution. Make stack implement the `Iterable` interface.

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Iterators

Q. What is an `Iterable` ?

A. Has a method that returns an `Iterator`.

Iterable interface

```
public interface Iterable<Item>
{
    Iterator<Item> iterator();
}
```

Q. What is an `Iterator` ?

A. Has methods `hasNext()` and `next()`.

Iterator interface

```
public interface Iterator<Item>
{
    boolean hasNext();
    Item next();
    void remove(); ← optional; use at your own risk
}
```

Q. Why make data structures `Iterable` ?

A. Java supports elegant client code.

"foreach" statement

```
for (String s : stack)
    StdOut.println(s);
```

equivalent code

```
Iterator<String> i = stack.iterator();
while (i.hasNext())
{
    String s = i.next();
    StdOut.println(s);
}
```

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Stack iterator: linked-list implementation

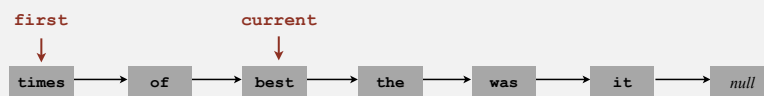
```
import java.util.Iterator;

public class Stack<Item> implements Iterable<Item>
{
    ...

    public Iterator<Item> iterator() { return new ListIterator(); }

    private class ListIterator implements Iterator<Item>
    {
        private Node current = first;

        public boolean hasNext() { return current != null; }
        public void remove() { /* not supported */ }
        public Item next()
        {
            Item item = current.item;
            current = current.next;
            return item;
        }
    }
}
```



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Stack iterator: array implementation

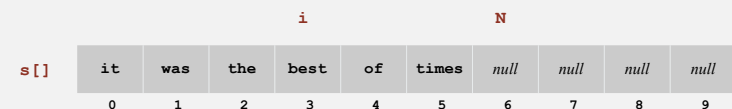
```
import java.util.Iterator;

public class Stack<Item> implements Iterable<Item>
{
    ...

    public Iterator<Item> iterator()
    { return new ReverseArrayIterator(); }

    private class ReverseArrayIterator implements Iterator<Item>
    {
        private int i = N;

        public boolean hasNext() { return i > 0; }
        public void remove() { /* not supported */ }
        public Item next() { return s[--i]; }
    }
}
```



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Iteration: concurrent modification

Q. What if client modifies the data structure while iterating?

A. A fail-fast iterator throws a `ConcurrentModificationException`.

concurrent modification

```
for (String s : stack)
    stack.push(s);
```

To detect:

- Count total number of `push()` and `pop()` operations in `stack`.
- Save current count in `*Iterator` subclass upon creation.
- Check that two values are still equal when calling `next()` and `hasNext()`.

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Bag API

Main application. Adding items to a collection and iterating (when order doesn't matter).

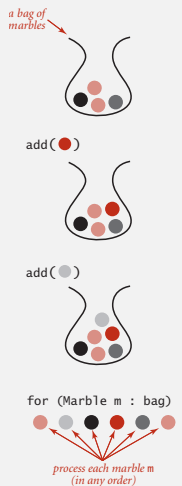
```
public class Bag<Item> implements Iterable<Item>

    Bag()                create an empty bag

    void add(Item x)     insert a new item onto bag

    int size()           number of items in bag

    Iterable<Item> iterator()  iterator for all items in bag
```



Implementation. Stack (without pop) or queue (without dequeue).

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- › stacks
- › resizing arrays
- › queues
- › generics
- › iterators
- › applications

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Java collections library

List interface. `java.util.List` is API for ordered collection of items.

```
public interface List<Item> implements Iterable<Item>

    List()                create an empty list

    boolean isEmpty()     is the list empty?

    int size()            number of items

    void add(Item item)   append item to the end

    Item get(int index)   return item at given index

    Item remove(int index) return and delete item at given index

    boolean contains(Item item) does the list contain the given item?

    Iterator<Item> iterator()  iterator over all items in the list

    ...
```

Implementations. `java.util.ArrayList` uses resizing array;
`java.util.LinkedList` uses linked list.

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Java collections library

`java.util.Stack`.

- Supports `push()`, `pop()`, `size()`, `isEmpty()`, and iteration.
- Also implements `java.util.List` interface from previous slide, including `get()`, `remove()`, and `contains()`.
- Bloating and poorly-designed API (why?) \Rightarrow don't use.

`java.util.Queue`. An interface, not an implementation of a queue.

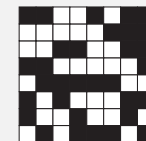
Best practices. Use our implementations of `stack`, `Queue`, and `Bag`.

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War story (from COS 226)

Generate random open sites in an N -by- N percolation system.

- Jenny: pick (i, j) at random; if already open, repeat.
Takes $\sim c_1 N^2$ seconds.
- Kenny: create a `java.util.LinkedList` of N^2 closed sites.
Pick an index at random and delete.
Takes $\sim c_2 N^4$ seconds.



Why is my program so slow?



Kenny

Lesson. Don't use a library until you understand its API!

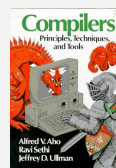
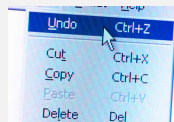
This course. Can't use a library until we've implemented it in class.



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Stack applications

- Parsing in a compiler.
- Java virtual machine.
- Undo in a word processor.
- Back button in a Web browser.
- PostScript language for printers.
- Implementing function calls in a compiler.
- ...



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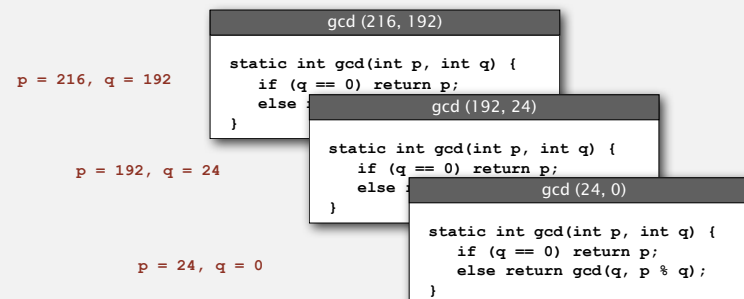
Function calls

How a compiler implements a function.

- Function call: **push** local environment and return address.
- Return: **pop** return address and local environment.

Recursive function. Function that calls itself.

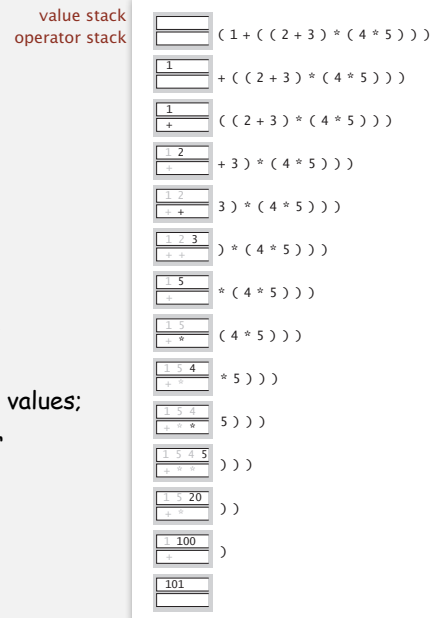
Note. Can always use an explicit stack to remove recursion.



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Arithmetic expression evaluation

Goal. Evaluate infix expressions.



Two-stack algorithm. [E. W. Dijkstra]

- Value: push onto the value stack.
- Operator: push onto the operator stack.
- Left parenthesis: ignore.
- Right parenthesis: pop operator and two values; push the result of applying that operator to those values onto the operand stack.

Context. An interpreter!

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Arithmetic expression evaluation demo

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Arithmetic expression evaluation

```
public class Evaluate
{
    public static void main(String[] args)
    {
        Stack<String> ops = new Stack<String>();
        Stack<Double> vals = new Stack<Double>();
        while (!StdIn.isEmpty()) {
            String s = StdIn.readString();
            if (s.equals("("))
            else if (s.equals("+")) ops.push(s);
            else if (s.equals("*")) ops.push(s);
            else if (s.equals(")")
            {
                String op = ops.pop();
                if (op.equals("+")) vals.push(vals.pop() + vals.pop());
                else if (op.equals("*")) vals.push(vals.pop() * vals.pop());
            }
            else vals.push(Double.parseDouble(s));
        }
        StdOut.println(vals.pop());
    }
}
```

```
% java Evaluate
( 1 + ( ( 2 + 3 ) * ( 4 * 5 ) ) )
101.0
```

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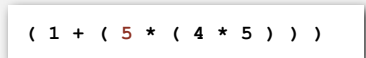
Correctness

Q. Why correct?

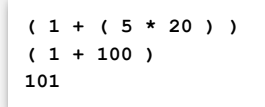
A. When algorithm encounters an operator surrounded by two values within parentheses, it leaves the result on the value stack.



as if the original input were:



Repeating the argument:



Extensions. More ops, precedence order, associativity.

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Stack-based programming languages

Observation 1. The 2-stack algorithm computes the same value if the operator occurs **after** the two values.

```
( 1 ( ( 2 3 + ) ( 4 5 * ) * ) + )
```

Observation 2. All of the parentheses are redundant!

```
1 2 3 + 4 5 * * +
```



Jan Lukasiewicz

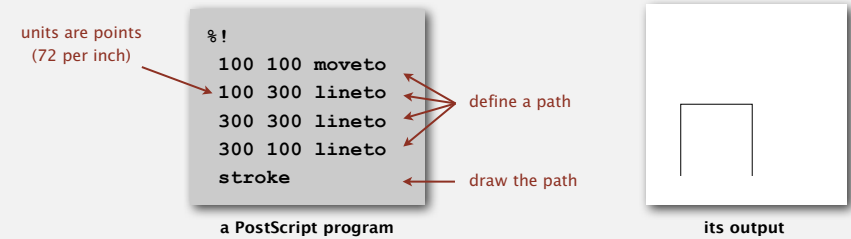
Bottom line. Postfix or "reverse Polish" notation.

Applications. Postscript, Forth, calculators, Java virtual machine, ...

PostScript

PostScript. [Warnock-Geschke 1980s]

- Postfix program code.
- Turtle graphics commands.
- Variables, types, text, loops, conditionals, functions, ...



Simple virtual machine, but not a toy.

- Easy to specify published page.
- Easy to implement in printers.
- Revolutionized the publishing world.



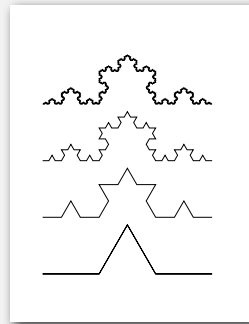
PostScript applications

Algorithms, 3rd edition. Figures created directly in PostScript.

```
%!
72 72 translate

/kochR
{
  2 copy ge { dup 0 rlineto }
  {
    3 div
    2 copy kochR 60 rotate
    2 copy kochR -120 rotate
    2 copy kochR 60 rotate
    2 copy kochR
  } ifelse
  pop pop
} def

0 0 moveto 81 243 kochR
0 81 moveto 27 243 kochR
0 162 moveto 9 243 kochR
0 243 moveto 1 243 kochR
stroke
```



see page 218

Algorithms, 4th edition. Figures created using enhanced version of `stdDraw` that saves to PostScript for vector graphics.



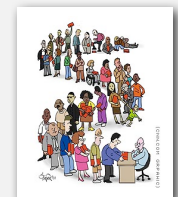
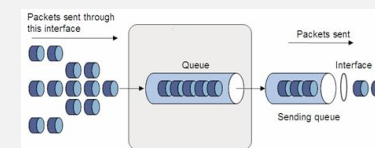
Queue applications

Familiar applications.

- iTunes playlist.
- Data buffers (iPod, TiVo).
- Asynchronous data transfer (file IO, pipes, sockets).
- Dispensing requests on a shared resource (printer, processor).

Simulations of the real world.

- Traffic analysis.
- Waiting times of customers at call center.
- Determining number of cashiers to have at a supermarket.

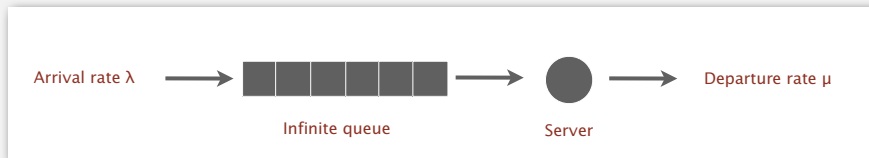


M/M/1 queuing model

M/M/1 queue.

- Customers arrive according to **Poisson process** at rate of λ per minute.
- Customers are serviced with rate of μ per minute.

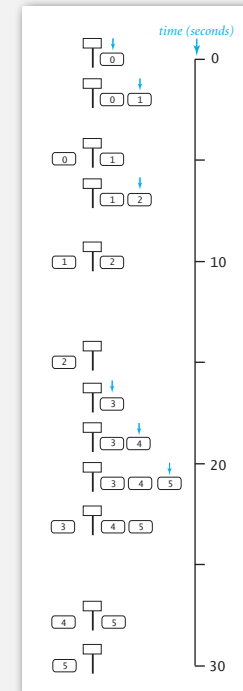
interarrival time has exponential distribution $\Pr[X \leq x] = 1 - e^{-\lambda x}$
 service time has exponential distribution $\Pr[X \leq x] = 1 - e^{-\mu x}$



- Q. What is average wait time W of a customer in system?
- Q. What is average number of customers L in system?

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M/M/1 queuing model: example simulation



	arrival	departure	wait
0	0	5	5
1	2	10	8
2	7	15	8
3	17	23	6
4	19	28	9
5	21	30	9

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M/M/1 queuing model: event-based simulation

```
public class MM1Queue
{
    public static void main(String[] args) {
        double lambda = Double.parseDouble(args[0]); // arrival rate
        double mu = Double.parseDouble(args[1]); // service rate
        double nextArrival = StdRandom.exp(lambda);
        double nextService = nextArrival + StdRandom.exp(mu);

        Queue<Double> queue = new Queue<Double>(); // queue of arrival times
        Histogram hist = new Histogram("M/M/1 Queue", 60);

        while (true)
        {
            while (nextArrival < nextService) // next event is an arrival
            {
                queue.enqueue(nextArrival);
                nextArrival += StdRandom.exp(lambda);
            }

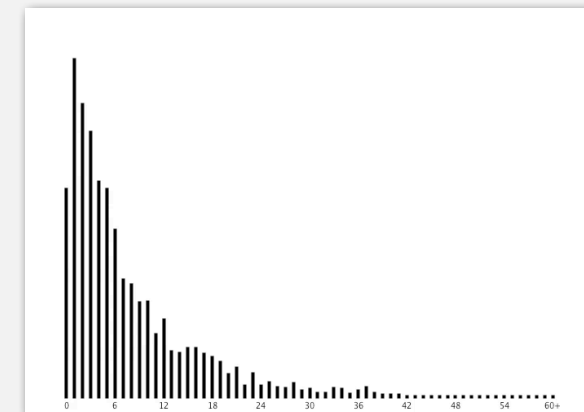
            double arrival = queue.dequeue(); // next event is a service completion
            double wait = nextService - arrival;
            hist.addDataPoint(Math.min(60, (int) (Math.round(wait))));
            if (queue.isEmpty()) nextService = nextArrival + StdRandom.exp(mu);
            else nextService = nextService + StdRandom.exp(mu);
        }
    }
}
```

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M/M/1 queuing model: experiments

Observation. If service rate μ is much larger than arrival rate λ , customers get good service.

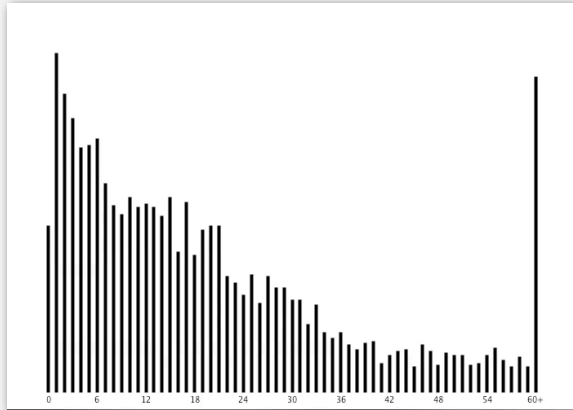
```
% java MM1Queue .2 .333
```



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Observation. As service rate μ approaches arrival rate λ , services goes to h^{***} .

```
% java MM1Queue .2 .25
```



Observation. As service rate μ approaches arrival rate λ , services goes to h^{***} .

```
% java MM1Queue .2 .21
```

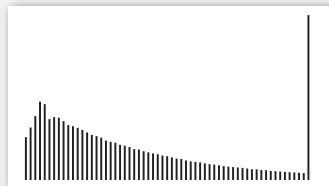
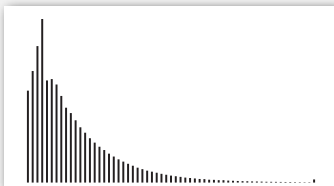


M/M/1 queue. Exact formulas known.

wait time W and queue length L approach infinity
as service rate approaches arrival rate

Little's Law

$$W = \frac{1}{\mu - \lambda}, \quad L = \lambda W$$



More complicated queuing models. Event-based simulation essential!
Queuing theory. See ORF 309.