Shape Analysis Basics

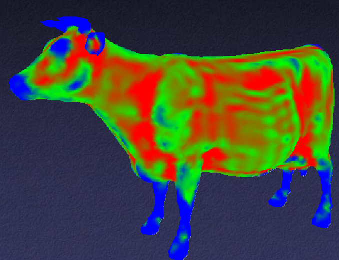
COS 526, Fall 2010

Some slides from Rusinkiewicz

Outline

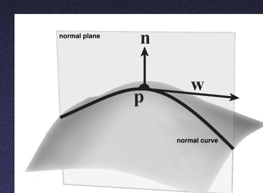
- Curvature
- PCA
- Distance
- Features

Curvature



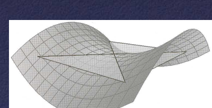
Curvature

- Curvature κ of a curve is reciprocal of radius of circle that best approximates it
- Defined at a point **p** in a direction **w**
- *Line has* $\kappa = 0$

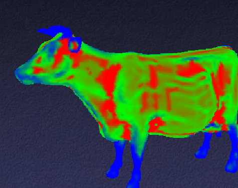


Principal Curvatures

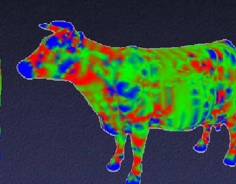
- The curvature at a point varies between some minimum and maximum – these are the *principal curvatures* κ_1 and κ_2
- They occur in the *principal directions* d_1 and d_2 which are perpendicular to each other



Principal Curvatures

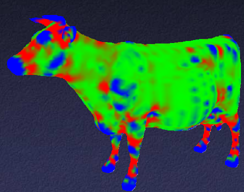


Minimum Curvature κ_1

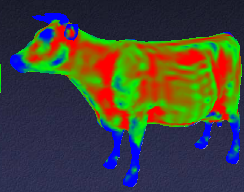


Maximum Curvature κ_2

Gaussian and Mean Curvature



Gauss Curvature
$K = \kappa_1 \kappa_2$



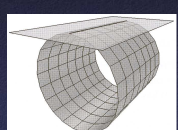
Mean Curvature
$H = ½ (\kappa_1 + \kappa_2)$

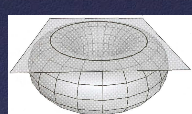
What Does Curvature Tell Us?

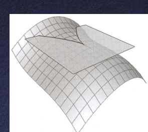
- Planar points:
  - Zero Gaussian curvature and zero mean curvature
  - Tangent plane intersects surface at infinity points

What Does Curvature Tell Us?

- Parabolic points:
  - Zero Gaussian curvature, non-zero mean curvature
  - Tangent plane intersects surface along 1 curves

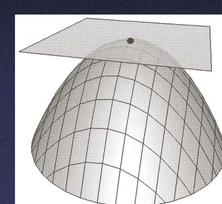






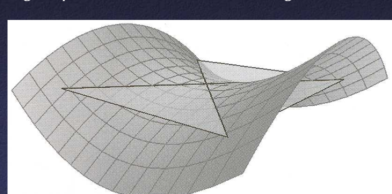
What Does Curvature Tell Us?

- Elliptical points:
  - Positive Gaussian curvature
  - Convex/concave depending on sign of mean curvature
  - Tangent plane intersects surface at 1 point



What Does Curvature Tell Us?

- Hyperbolic points:
  - Negative Gaussian curvature
  - Tangent plane intersects surface along 2 curves



What Does Curvature Tell Us?

- Mesh Saliency:
  - Motivated by models of perceptual salience
  - Difference between mean curvature blurred with σ and blurred with 2σ

Lee05

Outline

- Curvature
- PCA
- Distance
- Features

Principal Component Analysis (PCA)

- Tensor voting
  - Extract points $\{q_i\}$ in neighborhood
  - Compute covariance matrix M
  - Analyze eigenvalues and eigenvectors of M (via SVD)
  - Eigenvectors are Principal Axes

$$\mathbf{M} = \frac{1}{n}\sum_{i=1}^{n}\begin{bmatrix} q_i^x q_i^x & q_i^x q_i^y & q_i^x q_i^z \\ q_i^y q_i^x & q_i^y q_i^y & q_i^y q_i^z \\ q_i^z q_i^x & q_i^z q_i^y & q_i^z q_i^z \end{bmatrix}$$

Covariance Matrix

$$\mathbf{M} = \mathbf{USU}^t$$

$$\mathbf{S} = \begin{bmatrix} \lambda_a & 0 & 0 \\ 0 & \lambda_b & 0 \\ 0 & 0 & \lambda_c \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{bmatrix}$$

Eigenvalues & Eigenvectors

Principal Component Analysis (PCA)

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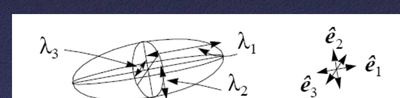
Eigenvalues & Eigenvectors

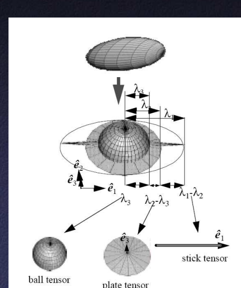
What Does PCA Tell Us?

- Eigenvectors are “Principal Axes of Inertia”
- Eigenvalues are variances of the point distribution in those directions



What Does PCA Tell Us?

- Helps differentiate nearly plane-like, from stick-like, from sphere-like, etc.



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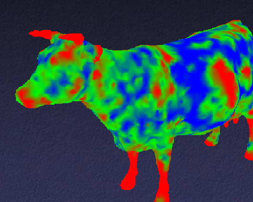
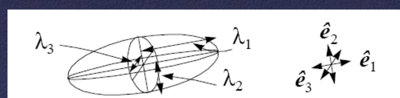


$\lambda_2 / (\lambda_1 + \lambda_2 + \lambda_3)$

What Does PCA Tell Us?

- Helps us construct a local coordinate frame for every point
  - Map $\hat{e}_1$ to X axis
  - Map $\hat{e}_2$ to Y axis
  - Map $\hat{e}_3$ to Z axis



Outline

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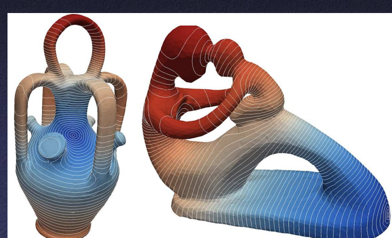
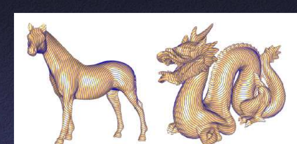
Distances on Meshes

- How close are two points p and q?



Distances on Meshes

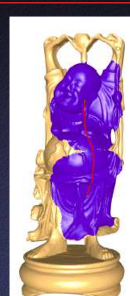
- Desirable properties:
  - Parameter-free
  - Is a metric
  - Smooth
  - Locally isotropic
  - Fast to compute
  - Shape aware
  - Insensitive to noise
  - Insensitive to topology changes



Surazhsky05

Geodesic Distance

- Length of shortest path between p and q on surface
- Can be computed exactly in $O(n^2 \log n)$ [Mitchell87]
- Often approximated with Dijkstra's algorithm on vertex graph



Surazhsky05

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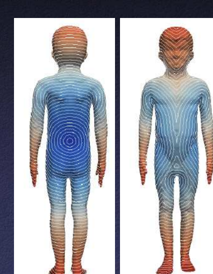
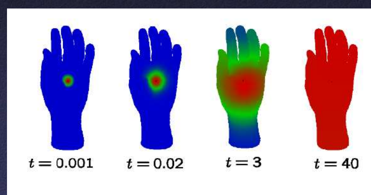


Diffusion Distance

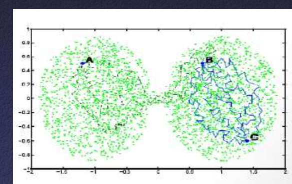
- Amount of heat that diffuses from p to q in time t



Sun09

Diffusion Distance

- Related to probability of random walk starting at p arriving at q after time t
  - Affected by all paths from p to q



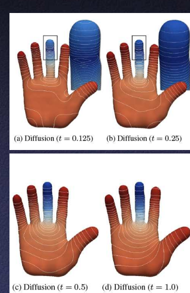
Lafon / Sun09

Diffusion Distance

- Can be computed by eigenanalysis of Laplacian

$$d_D(x,y)^2 = \sum_{k=1}^{\infty} e^{-2t\lambda_k} \left(\phi_k(x) - \phi_k(y)\right)^2$$

- Related to Euclidean distance in Spectral embedding
- Can be approximated for long times by first few eigenfunctions



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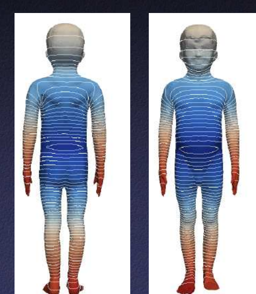

Diffusion Distance

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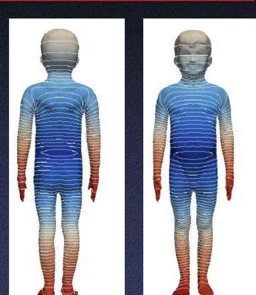

Biharmonic Distance

- Related to solution to biharmonic equations
- Can be computed by eigenanalysis of Laplacian

$$d_B(x,y)^2 = \sum_{k=1}^{\infty} \frac{\left(\phi_k(x) - \phi_k(y)\right)^2}{\lambda_k^2}.$$

Biharmonic Distance

- Properties:
  - Parameter-free
  - Is a metric
  - Smooth
  - Locally isotropic
  - Fast to compute
  - Shape aware
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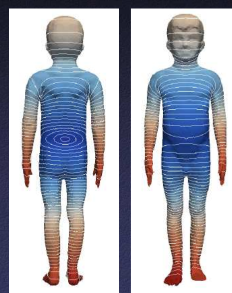
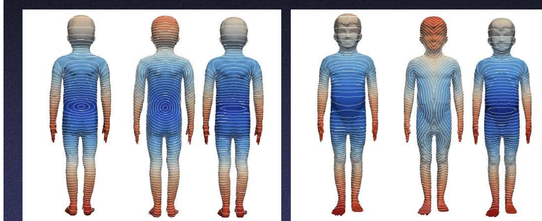


Distance Comparison



Biharmonic Geodesic Diffusion Biharmonic Geodesic Diffusion

Distance Comparison

Biharmonic Geodesic Diffusion

Distance Comparison

Biharmonic Geodesic Diffusion

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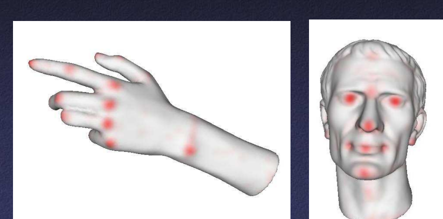
Features

Definition (Merriam-Webster)"

- "a prominent part or characteristic"



Point Features

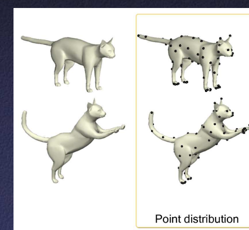
Applications:
- Maintaining shape features as process mesh
- Matching shape features as align meshes
- Reasoning about part decomposition
- Selecting viewpoints
- Visualization
- etc.

Point Feature Detection

Algorithmic methods

Iteratively choose furthest point

Others?



Point Feature Detection

Analytic methods

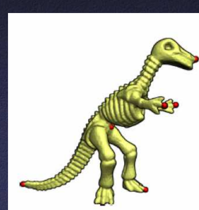
Extrema of DoG of curvature (saliency)

Extrema of Gauss curvature

Extrema of HKS

Extrema of AGD

etc.

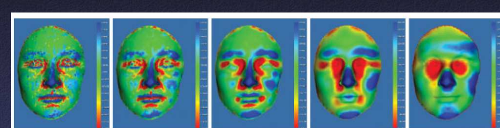


Zhang08

Point Feature Detection

Analytic methods

Many methods consider scale-space persistence



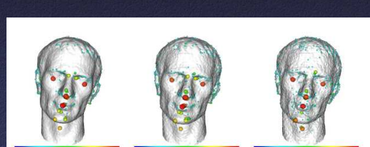
Zou08

Point Feature Detection

Analytic methods

Many methods consider scale-space persistence

Provides robustness to noise



Zou09

Point Feature Detection

Still difficult to detect "semantic points"

Zou09

Point Feature Detection

Your assignment …