PDE Stability Analysis

COS 323
Lax Equivalence Theorem

- For a well-posed linear PDE, necessary and sufficient conditions for solver to converge:
  - Consistency: local truncation error goes to zero
  - Stability: solution remains bounded

- Consistency derived from soundness of approximation to derivatives as $\Delta t \to 0$

- Stability: exact analysis often difficult
Von Neumann Stability Analysis

- Valid under assumptions (linear PDE, periodic boundary conditions), but often good starting point
- Fourier expansion (!) of solution
  \[ u(x, t) = \sum a_k (n \Delta t) e^{ikj \Delta x} \]
- Assume
  - Valid for linear PDEs, otherwise locally valid
  - Will be stable if magnitude of \( \xi \) is less than 1: errors decay, not grow, over time
Von Neumann: Diffusion Equation, FTCS

\[ u_t = ku_{xx} \]

\[ \frac{u_j^{n+1} - u_j^n}{\Delta t} = k \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} \]

\[ \frac{\xi - 1}{\Delta t} = k \frac{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}{(\Delta x)^2} \]

\[ \xi = 1 + \frac{k\Delta t}{(\Delta x)^2} (2 \cos k\Delta x - 2) \]

\[ \xi = 1 - \frac{4k\Delta t}{(\Delta x)^2} \sin^2 (k\Delta x / 2) \]
Von Neumann: Diffusion Equation, FTCS

• Since \(\sin^2\) between 0 and 1, to have amplitude less than 1 we need

\[
1 - \frac{4k \Delta t}{(\Delta x)^2} > -1
\]

\[
\Delta t < \frac{(\Delta x)^2}{2k}
\]

• Scheme is conditionally stable
Von Neumann: Advection Equation, FTCS

\[ u_t = -\nu u_x \]

\[ \frac{u_j^{n+1} - u_j^n}{\Delta t} = -\nu \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \]

\[ \xi - 1 = -\nu \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x} \]

\[ \xi = 1 - \frac{i\nu \Delta t}{\Delta x} \sin k \Delta x \]

- Magnitude always \( \geq 1 \): unconditionally unstable