Sampling and Aliasing

COS 323
Signal Processing

• Sampling a continuous function
• Convolve with reconstruction filter to re-create signal
How to Sample?

- Reconstructed signal might be very different from original: “aliasing”
Why Does Aliasing Happen?

• Sampling = multiplication by shah function $\text{III}(x)$ (also known as impulse train)
Digression: Delta Function

Kronecker delta

\[ \delta_{ij} = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } i = j
\end{cases} \]

Dirac delta

\[ \delta(x) = \begin{cases} 
0 & \text{if } x \neq 0 \\
\infty & \text{if } x = 0
\end{cases} \]

\[ \int \delta(x) \, dx = 1 \]

- Can think of as \[ \delta(x) = \lim_{\sigma \to 0^+} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \]
Scaled and Translated Dirac Delta

\[ c \delta(x - x_0) = \begin{cases} 0 & \text{if } x \neq x_0 \\ \infty & \text{if } x = x_0 \end{cases} \]

\[ \int c \delta(x - x_0) dx = c \]
Impulse Train

$$\text{III}(x) = \cdots + \partial(x + 2) + \partial(x + 1) + \partial(x) + \partial(x - 1) + \partial(x - 2) + \cdots$$
Why Does Aliasing Happen?

- Sampling = multiplication by shah function $\text{III}(x)$ (also known as impulse train)

![Diagram showing the process of sampling a function and the resulting aliased signal](image)
Fourier Analysis

- Multiplication in primal space = convolution in frequency space

\[ F(f(x)g(x)) = F(f(x)) \ast F(g(x)) \]

- Fourier transform of \( III \) is \( III \)
Fourier Analysis

• Result: high frequencies can “alias” into low frequencies
Aliasing
• Convolution with reconstruction filter = multiplication in frequency space

\[ F(f(x) \ast g(x)) = F(f(x))F(g(x)) \]
• Conclusions:
  – High frequencies can alias into low frequencies
  – Can’t be cured by a different reconstruction filter
  – Nyquist limit: capture all frequencies iff bandlimited – maximum frequency < ½ sampling rate
Aliasing strikes!
Other Aliasing Examples

- Car wheel “spins backwards” on film
- Jaggies in graphics
- “Crawling jaggies” on edges of objects as they move
Filters for Sampling

- **Solution**: insert filter *before* sampling
  - “Sampling” or “bandlimiting” or “antialiasing” filter
    - Low-pass filter
    - Eliminate frequency content above Nyquist limit
    - Result: aliasing replaced by blur
    - Partial alternative: *oversampling*, digital filtering
Antialiasing Jaggies

Aliased

Postfiltered: blurry jaggies

Correctly prefiltered
Ideal Sampling Filter

- **“Brick wall” filter:**
  box in frequency

- **In space: sinc function**
  - $sinc(x) = \sin(x) / x$
  - Infinite support
  - Possibility of “ringing”
Cheap Sampling Filter

- **Box in space**
  - Cheap to evaluate
  - Finite support

- **In frequency: sinc**
  - Imperfect bandlimiting
Gaussian Sampling Filter

• Fourier transform of Gaussian = Gaussian

• Good compromise as sampling filter:
  – Well approximated by function w. finite support
  – Good bandlimiting performance