Constrained Optimization

COS 323

Linear Objective, Linear Constraints

Standard form: maximize objective

$$\zeta = c_1 x_1 + c_2 x_2 + \cdots$$

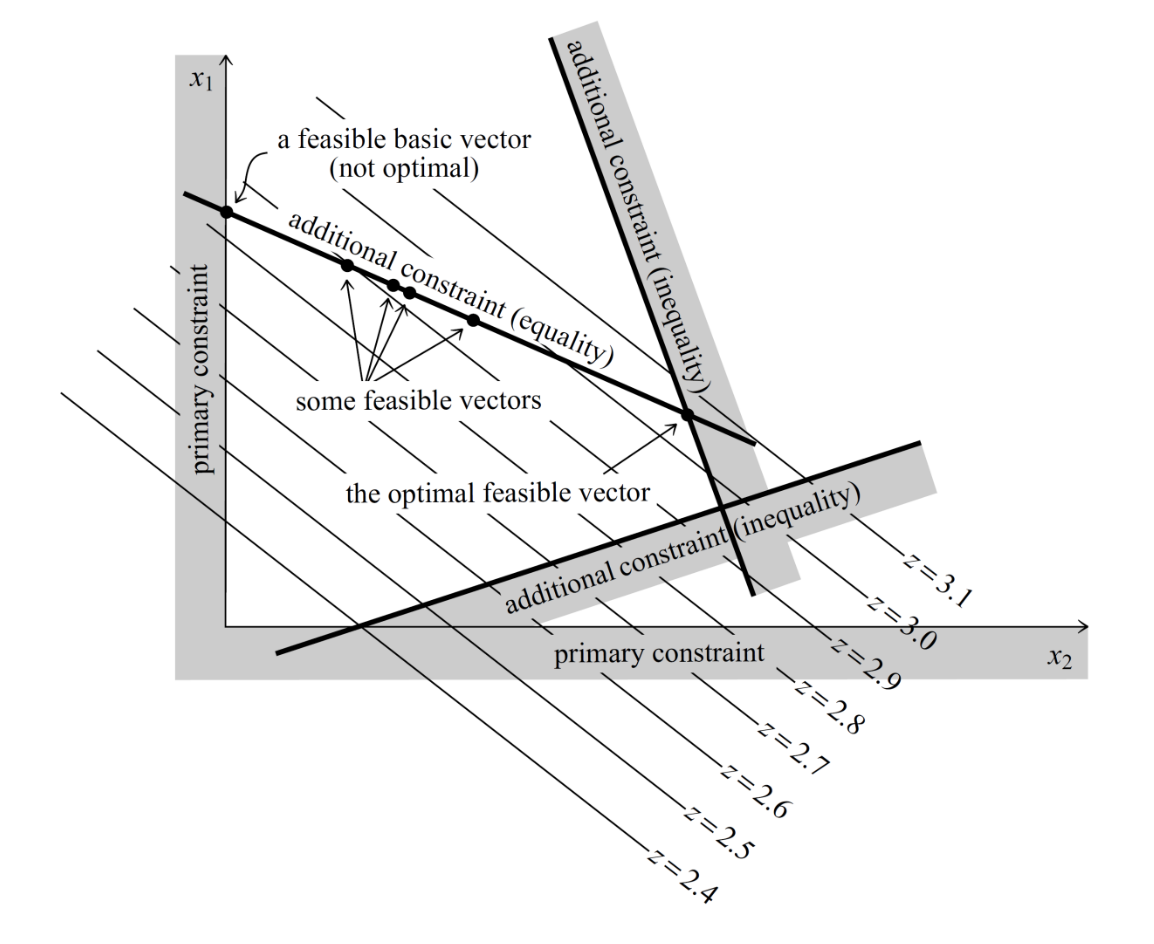
with primary constraints

$$x_1 \geq 0,\ x_2 \geq 0,\ \cdots$$

and additional constraints

$$a_{11} x_1 + a_{12}\, x_2 + \cdots \leq b_1$$
$$a_{21} x_1 + a_{22}\, x_2 + \cdots \leq b_2$$

Linear Programming



[Press et al.]

Simplex Method

- Slides from Robert Vanderbei

Simplex Method

- In theory: can take very long – exponential in the input length
- In practice: efficient – # of iterations typically a few times # of constraints
- There exist provably polynomial-time algorithms

General Optimization with Equality Constraints

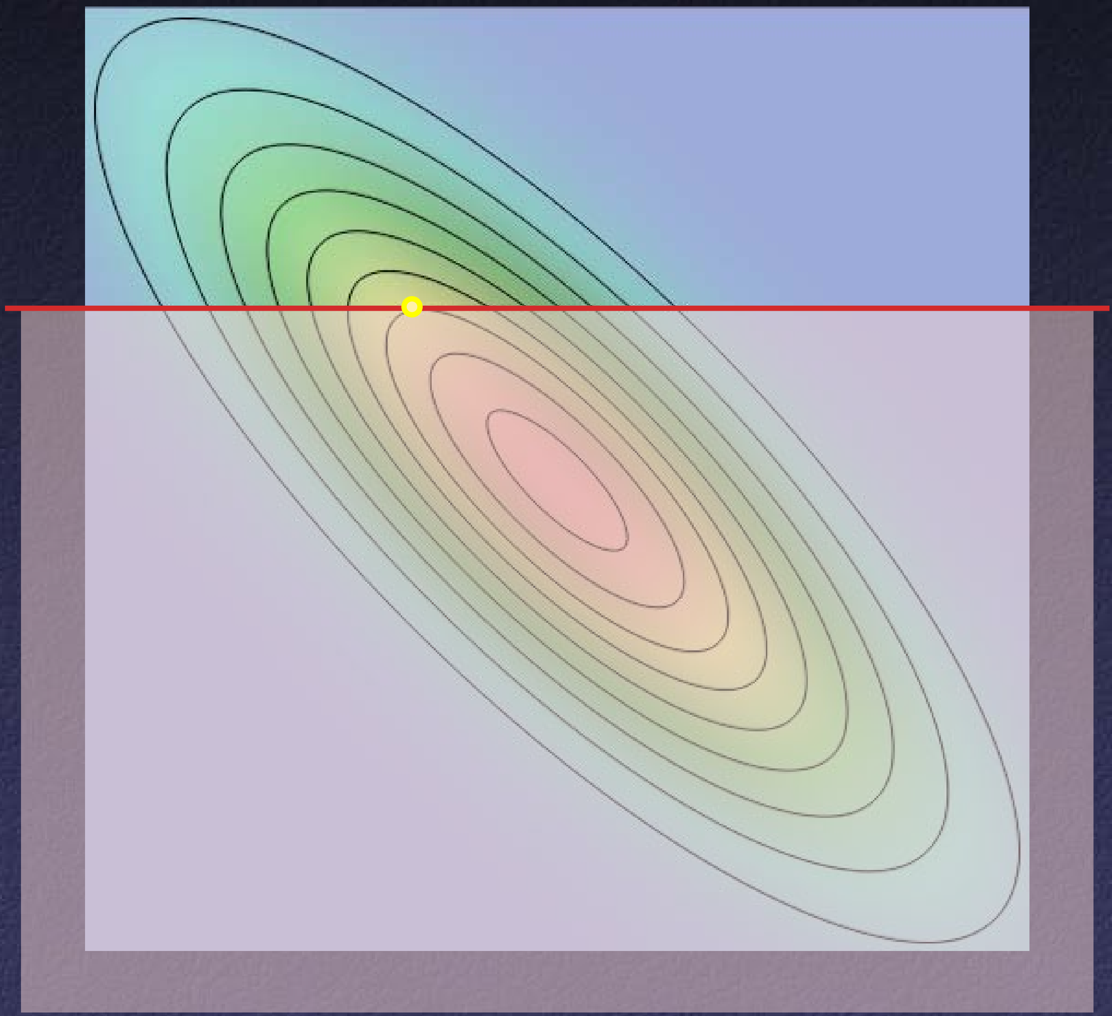
- Minimize $f(x)$ subject to $g_i(x) = 0$
- Method of Lagrange multipliers: convert to a higher-dimensional problem
- Minimize $f(x) + \sum \lambda_i g_i(x)$ w.r.t. $(x_1 \ldots x_n; \lambda_1 \ldots \lambda_k)$

General Optimization with Inequality Constraints

- Minimize $f(x)$ subject to $h_i(x) \le 0$
- Karush-Kuhn-Tucker (KKT) conditions

- Minimize $f(x) + \sum \lambda_i h_i(x)$ w.r.t. $(x_1 \ldots x_n; \lambda_1 \ldots \lambda_k)$
- Subject to $h_i(x) \le 0, \quad \lambda_i(x) \ge 0, \quad \lambda_i(x) h_i(x) = 0$

KKT Conditions



minimize $f(x)$

with $h(x) \leq 0$

1. $\partial/_{\partial x}(f(x)+\lambda h(x)) = 0$
2. $\partial/_{\partial \lambda}(f(x)+\lambda h(x)) = 0$
3. $h(x) \leq 0$
4. $\lambda \geq 0$
5. $\lambda\, h(x) = 0$

Quadratic Programming

- The KKT conditions allow writing a system with quadratic objective and linear constraints as a linear program
  - Solve with simplex, etc.