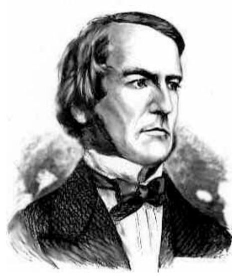
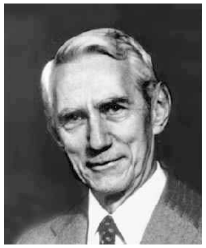
# 6. Combinational Circuits



George Boole (1815 - 1864)



Claude Shannon (1916 - 2001)

## Digital Circuits

Q. What is a digital system?
A. Digital: signals are 0 or 1.

analog: signals vary continuously

Q. Why digital systems?
A. Accurate, reliable, fast, cheap.

Basic abstractions.
- On, off.
- Wire: propagates on/off value.
- Switch: controls propagation of on/off values through wires.

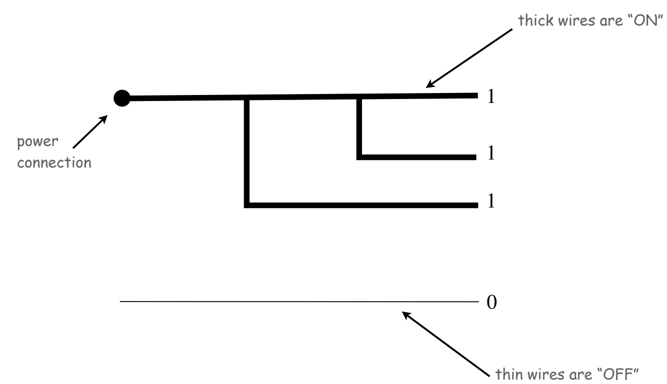
Applications. Cell phone, iPod, antilock brakes, microprocessors, …



# Building Blocks

## Wires

Wires.
- ON (1): connected to power.
- OFF (0): not connected to power.
- If a wire is connected to a wire that is on, that wire is also on.
- Typical drawing convention: "flow" from top, left to bottom, right.



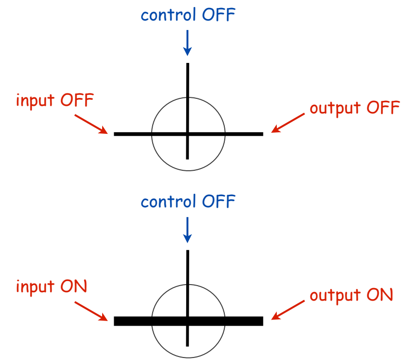
## Controlled Switch

Controlled switch.
- 3 connections:  input, output, control.



## Controlled Switch

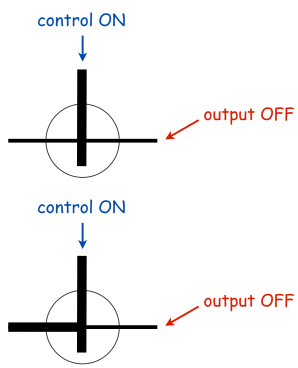
Controlled switch.
- 3 connections:  input, output, control.
- control OFF: output is connected to input




## Controlled Switch

Controlled switch.
- 3 connections:  input, output, control.

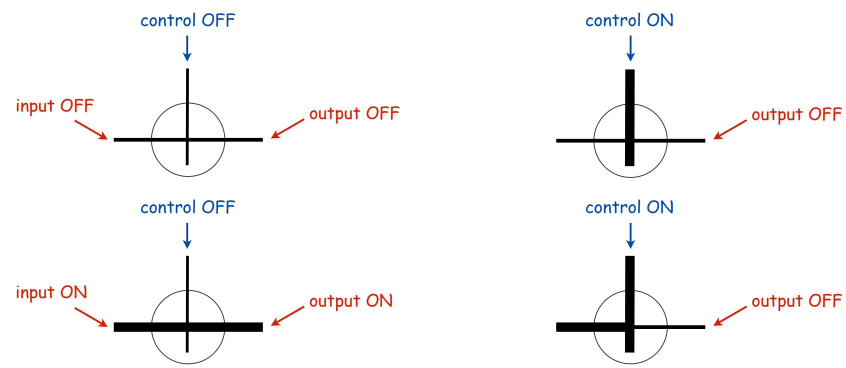
- control ON: output is disconnected from input






## Controlled Switch

Controlled switch.
- 3 connections:  input, output, control.
- control OFF: output is connected to input
- control ON: output is disconnected from input

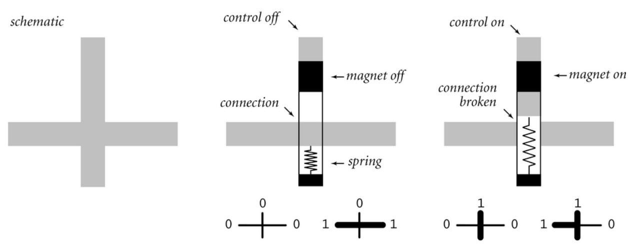



idealized model of "pass transistors" found in real integrated circuits

## Implementing a Controlled Switch

Relay implementation.

- 3 connections: input, output, control.
- Magnetic force pulls on a contact that cuts electrical flow.




*Anatomy of a relay (controlled switch)*



## First Level of Abstraction

Separates physical world from logical world.

- we assume that switches operate as specified
- that is the only assumption
- physical realization of switch is irrelevant to design

Physical realization dictates performance

- size
- speed
- power

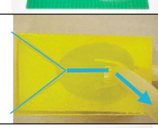
New technology immediately gives new computer.

Better switch? Better computer.



## Controlled Switches: A First Level of Abstraction

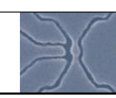
Some amusing attempts to prove the point:

| Technology | "Information" | Switch |
|---|---|---|
| pneumatic | air pressure |  |
| fluid | water pressure |  |
| relay | electric potential |  |



## Controlled Switches: A First Level of Abstraction

Real-world examples that prove the point:

| technology | switch |
|---|---|
| relay |  |
| vacuum tube |  |
| transistor |  |
| "pass transistor" in integrated circuit |  |
| atom-thick transistor |  |

## Controlled Switches: A First Level of Abstraction ?

VLSI = Very Large Scale Integration

Technology:
Deposit materials on substrate.

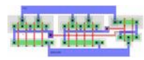
Key property:
Crossing lines are controlled switches.

Key challenge in physical world:
Fabricating physical circuits
with billions of controlled switches

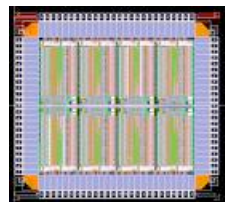
Key challenge in "abstract" world:
Understanding behavior of circuits
with billions of controlled switches

Bottom line: Circuit = Drawing (!)

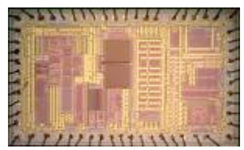
drawing



drawing

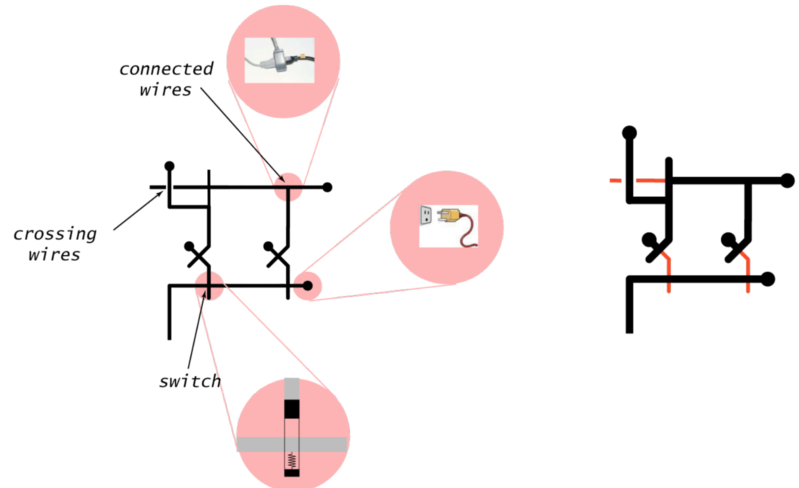


circuit





## Circuit Anatomy



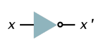

need more "levels of abstraction" to understand circuit behavior



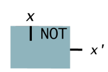
## Second Level of Abstraction: Logic Gates

***NOT = x'***

| x | NOT |
|---|---|
| 0 | 1 |
| 1 | 0 |



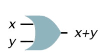

symbol



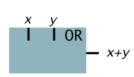

NOT gate

***OR = x+y***

| x y | OR |
|---|---|
| 0 0 | 0 |
| 0 1 | 1 |
| 1 0 | 1 |
| 1 1 | 1 |



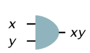

symbol



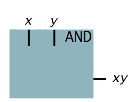

OR gate

***AND = xy***

| x y | AND |
|---|---|
| 0 0 | 0 |
| 0 1 | 0 |
| 1 0 | 0 |
| 1 1 | 1 |




symbol




AND gate



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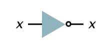
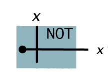
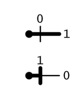



symbol

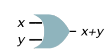



NOT gate

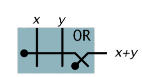



***OR = x+y***

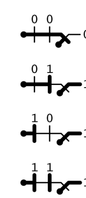
| x y | OR |
|---|---|
| 0 0 | 0 |
| 0 1 | 1 |
| 1 0 | 1 |
| 1 1 | 1 |




symbol

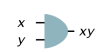



OR gate

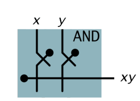



***AND = xy***

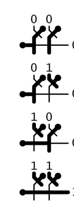
| x y | AND |
|---|---|
| 0 0 | 0 |
| 0 1 | 0 |
| 1 0 | 0 |
| 1 1 | 1 |




symbol



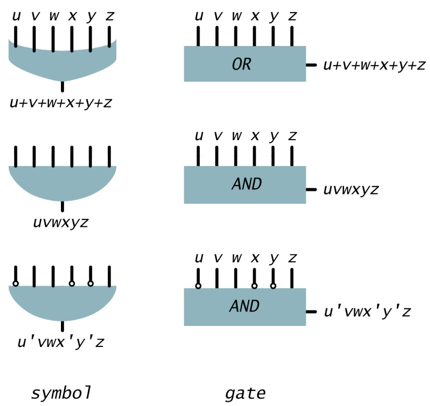

AND gate




*implementations with switches*

## Multiway Gates

Multiway gates.
- OR: 1 if any input is 1; 0 otherwise.
- AND: 1 if all inputs are 1; 0 otherwise.
- Generalized: negate some inputs.






## Multiway Gates

Multiway gates.
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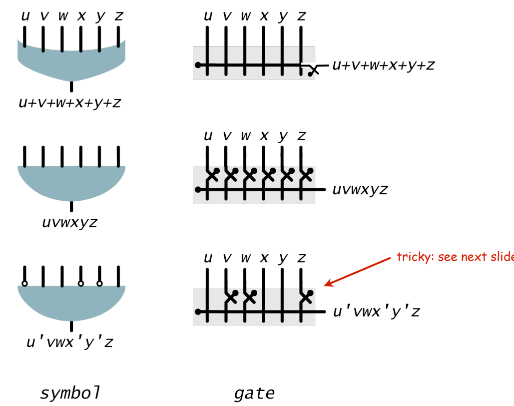




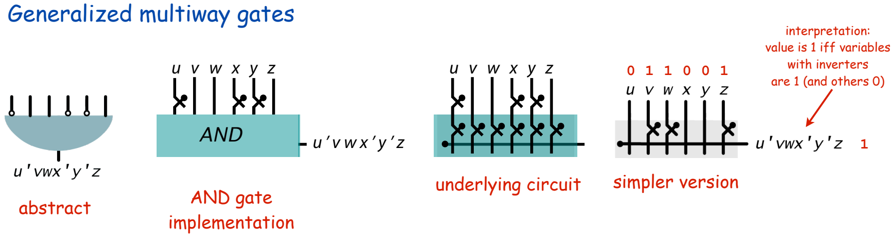

## Building blocks (summary)

Wires

Controlled switches

Gates

Generalized multiway gates






# Boolean Algebra

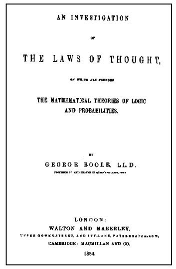
## Boolean Algebra

History.
- Developed by Boole to solve mathematical logic problems (1847).
- Shannon master's thesis applied it to digital circuits (1937).

*"possibly the most important, and also the most famous, master's thesis of the [20th] century" — Howard Gardner*

Boolean algebra.
- Boolean variable: value is 0 or 1.
- Boolean function: function whose inputs and outputs are 0, 1.

Relationship to circuits.
- Boolean variable: signal.
- Boolean function: circuit.


AN INVESTIGATION

OF

THE LAWS OF THOUGHT,

THE MATHEMATICAL THEORIES OF LOGIC AND PROBABILITIES.

GEORGE BOOLE, LL.D.

LONDON:
WALTON AND MABERLY,
UPPER GOWER-STREET, AND IVY-LANE, PATERNOSTER-ROW,
CAMBRIDGE: MACMILLAN AND CO.
1854.



## Boole Orders Lunch



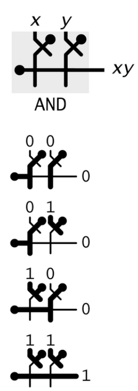



## Truth Table

Truth table.
- Systematic method to describe Boolean function.
- One row for each possible input combination.
- $n$ inputs $\Rightarrow$ $2^n$ rows.

| $x$ | $y$ | $x\,y$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

*AND* truth table






## Truth Table for Functions of 2 Variables

Truth table.
- 16 Boolean functions of 2 variables. ← every 4-bit value represents one

| *x* | *y* | *ZERO* | *AND* | | *x* | | *y* | *XOR* | *OR* |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |

*truth table for all Boolean functions of 2 variables*

| *x* | *y* | *NOR* | *EQ* | *y'* | | *x'* | | *NAND* | *ONE* |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |

*truth table for all Boolean functions of 2 variables*

## Truth Table for Functions of 3 Variables

Truth table.

- 16 Boolean functions of 2 variables. ⟵ every 4-bit value represents one
- 256 Boolean functions of 3 variables. ⟵ every 8-bit value represents one
- $2^{\wedge}(2^{\wedge}n)$ Boolean functions of $n$ variables! ⟵ every $2^n$-bit value represents one

| $x$ | $y$ | $z$ | $AND$ | $OR$ | $MAJ$ | $ODD$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 0 | 1 | 1 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

*some functions of 3 variables*



## Universality of AND, OR, NOT

Fact. Any Boolean function can be expressed using $AND$, $OR$, $NOT$.

- { $AND$, $OR$, $NOT$ } are universal.
- Ex: $XOR(x, y) = xy' + x'y$.

| notation | meaning |
|---|---|
| $x'$ | $NOT\ x$ |
| $x\ y$ | $x\ AND\ y$ |
| $x + y$ | $x\ OR\ y$ |

Expressing $XOR$ Using $AND$, $OR$, $NOT$

| $x$ | $y$ | $x'$ | $y'$ | $x'y$ | $xy'$ | $x'y + xy'$ | $x\ XOR\ y$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |

Exercise. Show $\{AND, NOT\}$, $\{OR, NOT\}$, $\{NAND\}$ are universal.
Hint. DeMorgan's law: $(x' y')' = x + y$.



## Sum-of-Products

Sum-of-products. Systematic procedure for representing a Boolean function using $AND$, $OR$, $NOT$.

- Form $AND$ term for each 1 in Boolean function.
- $OR$ terms together.

proves that { $AND$, $OR$, $NOT$ } are universal

| $x$ | $y$ | $z$ | $MAJ$ | $x'yz$ | $xy'z$ | $xyz'$ | $xyz$ | $x'yz + xy'z + xyz' + xyz$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |

expressing $MAJ$ using sum-of-products



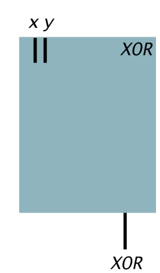
## Translate Boolean Formula to Boolean Circuit

Sum-of-products. $XOR$.

**XOR = x'y + xy'**

| x y | XOR |
|---|---|
| 0 0 | 0 |
| 0 1 | 1 |
| 1 0 | 1 |
| 1 1 | 0 |

Truth table

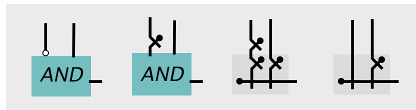



Circuit

## Translate Boolean Formula to Boolean Circuit

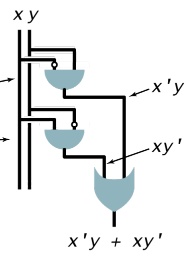
Sum-of-products. *XOR.*

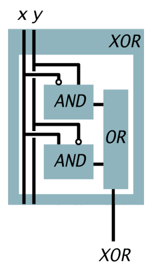
Key transformation from abstract to real circuit




***XOR = x'y + xy'***

| x y | XOR |
|---|---|
| 0 0 | 0 |
| 0 1 | 1 |
| 1 0 | 1 |
| 1 1 | 0 |



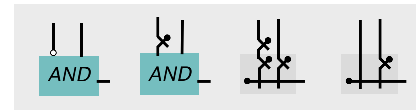

*Truth table* *Abstract circuit* *Circuit*



## Translate Boolean Formula to Boolean Circuit

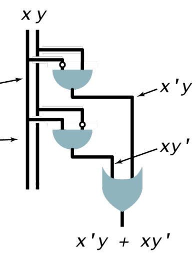
Example 1. *XOR.*

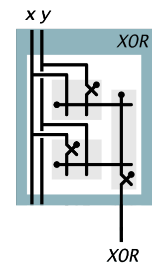
Key transformation from abstract to real circuit




***XOR = x'y + xy'***

| x y | XOR |
|---|---|
| 0 0 | 0 |
| 0 1 | 1 |
| 1 0 | 1 |
| 1 1 | 0 |




*Truth table* *Abstract circuit* *Circuit*

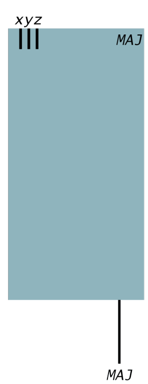


## Translate Boolean Formula to Boolean Circuit

Example 2. Majority.

***MAJ = x'yz + xy'z + xyz'+ xyz***

| x y z | MAJ |
|---|---|
| 0 0 0 | 0 |
| 0 0 1 | 0 |
| 0 1 0 | 0 |
| 0 1 1 | 1 |
| 1 0 0 | 0 |
| 1 0 1 | 1 |
| 1 1 0 | 1 |
| 1 1 1 | 1 |




*Truth table* *Circuit*

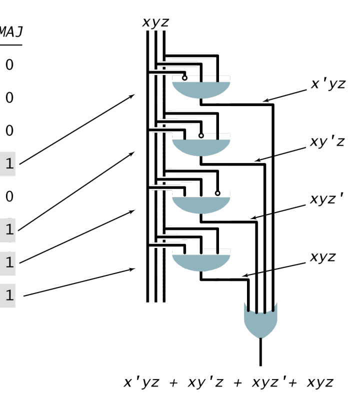


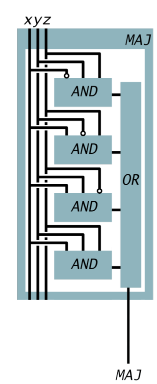
## Translate Boolean Formula to Boolean Circuit

Example 2. Majority.

***MAJ = x'yz + xy'z + xyz'+ xyz***

| x y z | MAJ |
|---|---|
| 0 0 0 | 0 |
| 0 0 1 | 0 |
| 0 1 0 | 0 |
| 0 1 1 | 1 |
| 1 0 0 | 0 |
| 1 0 1 | 1 |
| 1 1 0 | 1 |
| 1 1 1 | 1 |



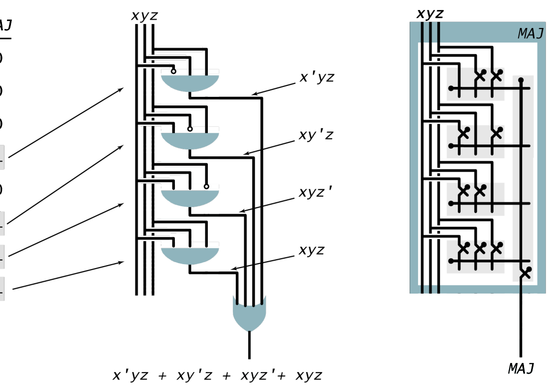

*Truth table* *Abstract circuit* *Circuit*

## Translate Boolean Formula to Boolean Circuit

Example 2. Majority.

**MAJ = x'yz + xy'z + xyz'+ xyz**

| x | y | z | MAJ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 |




*Truth table* *Abstract circuit* *Circuit*

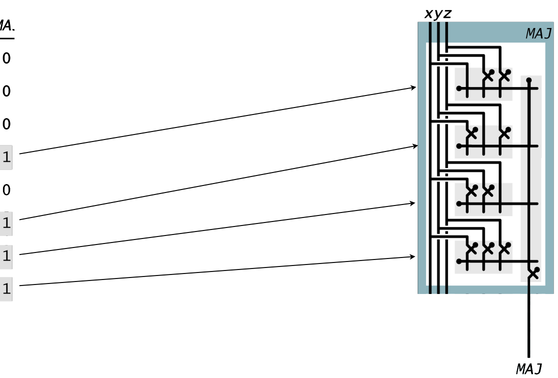


## Translate Boolean Formula to Boolean Circuit

Example 2. Majority.

**MAJ = x'yz + xy'z + xyz'+ xyz**

| x | y | z | MA. |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 |




*Truth table* *Abstract circuit* *Circuit*

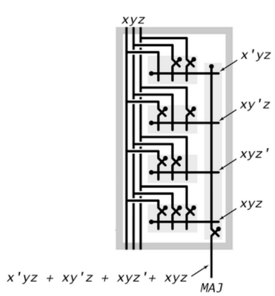


## Simplification Using Boolean Algebra

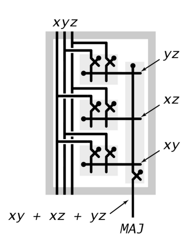
Many possible circuits for each Boolean function.

- Sum-of-products not necessarily optimal in:
  - number of switches (space)
  - depth of circuit (time)

Ex. $MAJ(x, y, z) = x'yz + xy'z + xyz' + xyz = xy + yz + xz$.




size = 10, depth = 2




size = 7, depth = 2



## Combinational Circuit Design: Summary

Problem: Compute the value of a boolean function

Ingredients.

- *AND* gates.
- *OR* gates.
- *NOT* gates.
- Wire.

Instructions.

- Step 1: represent input and output signals with Boolean variables.
- Step 2: construct truth table to carry out computation.
- Step 3: derive (simplified) Boolean expression using sum-of products.
- Step 4: transform Boolean expression into circuit.

Bottom line (profound idea):
It is easy to design a circuit to compute ANY boolean function.

Caveat (stay tuned): Circuit might be huge.

## Translate Boolean Formula to Boolean Circuit

Example 3. Odd parity
- 1 if odd number of inputs are 1.
- 0 otherwise.

| $x$ | $y$ | $z$ | $ODD$ | $x'y'z$ | $x'yz'$ | $xy'z'$ | $xyz$ | $x'y'z + x'yz' + xy'z' + xyz$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |

Expressing $ODD$ using sum-of-products

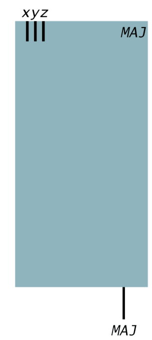


## Translate Boolean Formula to Boolean Circuit

Example 3. Odd parity
- 1 if odd number of inputs are 1.
- 0 otherwise.

MAJ = x'yz + xy'z + xyz'+ xyz

| x y z | MAJ |
|---|---|
| 0 0 0 | 0 |
| 0 0 1 | 0 |
| 0 1 0 | 0 |
| 0 1 1 | 1 |
| 1 0 0 | 0 |
| 1 0 1 | 1 |
| 1 1 0 | 1 |
| 1 1 1 | 1 |




ODD = x'y'z + x'yz'+ xy'z'+ xyz

| x y z | ODD |
|---|---|
| 0 0 0 | 0 |
| 0 0 1 | 1 |
| 0 1 0 | 1 |
| 0 1 1 | 0 |
| 1 0 0 | 1 |
| 1 0 1 | 0 |
| 1 1 0 | 0 |
| 1 1 1 | 1 |

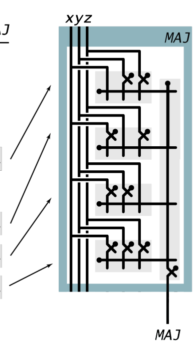



## Translate Boolean Formula to Boolean Circuit

Example 3. Odd parity
- 1 if odd number of inputs are 1.
- 0 otherwise.

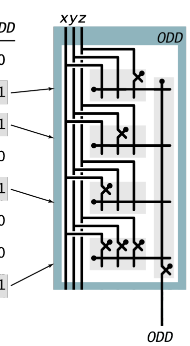
MAJ = x'yz + xy'z + xyz'+ xyz

| x y z | MAJ |
|---|---|
| 0 0 0 | 0 |
| 0 0 1 | 0 |
| 0 1 0 | 0 |
| 0 1 1 | 1 |
| 1 0 0 | 0 |
| 1 0 1 | 1 |
| 1 1 0 | 1 |
| 1 1 1 | 1 |




ODD = x'y'z + x'yz'+ xy'z'+ xyz

| x y z | ODD |
|---|---|
| 0 0 0 | 0 |
| 0 0 1 | 1 |
| 0 1 0 | 1 |
| 0 1 1 | 0 |
| 1 0 0 | 1 |
| 1 0 1 | 0 |
| 1 1 0 | 0 |
| 1 1 1 | 1 |






# Adder Circuit

## Let's Make an Adder Circuit

Goal. $x + y = z$ for 4-bit integers.

- We build 4-bit adder: 9 inputs, 4 outputs.
- Each output bit is a boolean function of the inputs.
- Standard method applies.

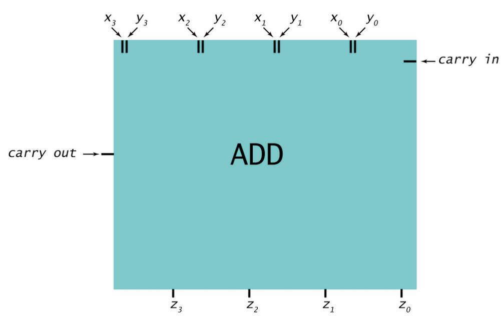
same idea scales to 64-bit adder in your computer

Step 1. Represent input and output in binary.

| | | | | |
|---|---|---|---|---|
| | 1 | 1 | 1 | 0 |
| | 2 | 4 | 8 | 7 |
| + | 3 | 5 | 7 | 9 |
| | 6 | 0 | 6 | 6 |

| | | | | |
|---|---|---|---|---|
| | 1 | 1 | 0 | 0 |
| | 0 | 0 | 1 | 0 |
| + | 0 | 1 | 1 | 1 |
| | 1 | 0 | 0 | 1 |

| | | | | |
|---|---|---|---|---|
| | $x_3$ | $x_2$ | $x_1$ | $x_0$ |
| + | $y_3$ | $y_2$ | $y_1$ | $y_0$ |
| | $z_3$ | $z_2$ | $z_1$ | $z_0$ |






## Let's Make an Adder Circuit

Goal. $x + y = z$ for 4-bit integers.

Step 2. [first attempt]

- Build truth table.

| | $c_{out}$ | | | $c_{in}$ |
|---|---|---|---|---|
| | $x_3$ | $x_2$ | $x_1$ | $x_0$ |
| + | $y_3$ | $y_2$ | $y_1$ | $y_0$ |
| | $z_3$ | $z_2$ | $z_1$ | $z_0$ |

*4-bit adder truth table*

| $c_0$ | $x_3$ | $x_2$ | $x_1$ | $x_0$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $z_3$ | $z_2$ | $z_1$ | $z_0$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| . | . | . | . | . | . | . | . | . | . | . | . | . |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

$2^{8+1} = 512$ rows!

Q. Why is this a bad idea?

A. 128-bit adder: $2^{256+1}$ rows >> # electrons in universe!



## Let's Make an Adder Circuit

Goal. $x + y = z$ for 4-bit integers.

Step 2. Do one bit at a time!

- Build truth table for carry bit.
- Build truth table for summand bit.

| | | | | |
|---|---|---|---|---|
| $c_{out}$ | $c_3$ | $c_2$ | $c_1$ | $c_0 = 0$ |
| | $x_3$ | $x_2$ | $x_1$ | $x_0$ |
| + | $y_3$ | $y_2$ | $y_1$ | $y_0$ |
| | $z_3$ | $z_2$ | $z_1$ | $z_0$ |

*carry bit*

| $x_i$ | $y_i$ | $c_i$ | $c_{i+1}$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 |

*summand bit*

| $x_i$ | $y_i$ | $c_i$ | $z_i$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |



## Let's Make an Adder Circuit

Goal. $x + y = z$ for 4-bit integers.

Step 3. A surprise!

- carry bit is majority function
- summand bit is odd parity function.

| | | | | |
|---|---|---|---|---|
| $c_{out}$ | $c_3$ | $c_2$ | $c_1$ | $c_0 = 0$ |
| | $x_3$ | $x_2$ | $x_1$ | $x_0$ |
| + | $y_3$ | $y_2$ | $y_1$ | $y_0$ |
| | $z_3$ | $z_2$ | $z_1$ | $z_0$ |

*carry bit*

| $x_i$ | $y_i$ | $c_i$ | $c_{i+1}$ | MAJ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 |

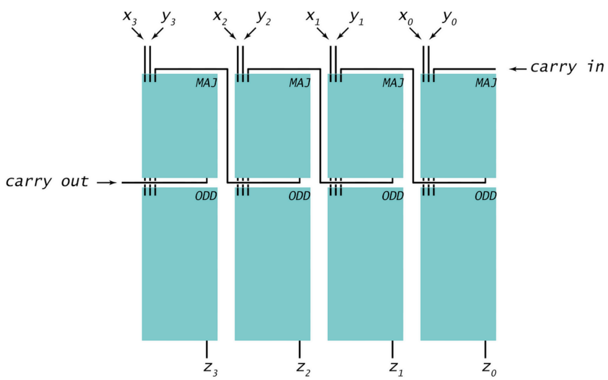
*summand bit*

| $x_i$ | $y_i$ | $c_i$ | $z_i$ | ODD |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 |

## Let's Make an Adder Circuit

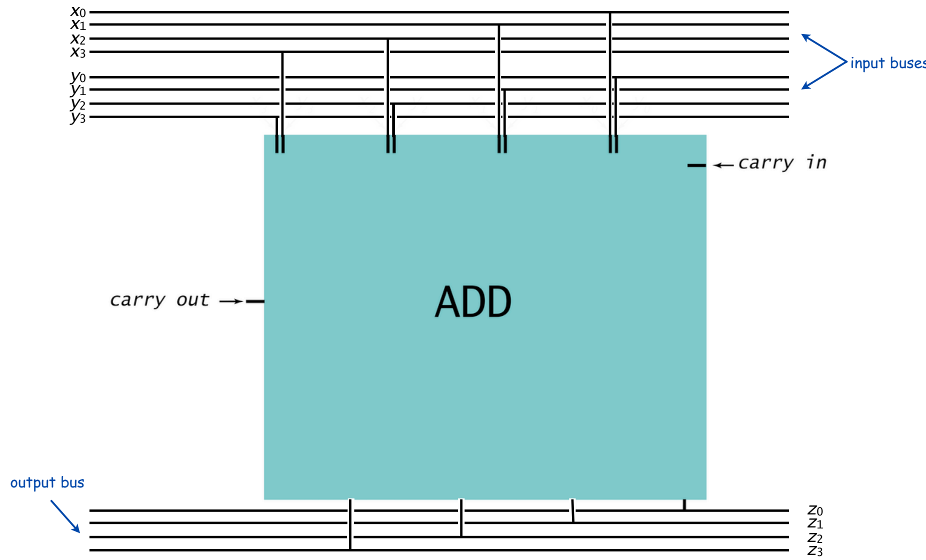
Goal. $x + y = z$ for 4-bit integers.

Step 4.
- Transform Boolean expression into circuit (use known components!).
- Chain together 1-bit adders.
- That's it!





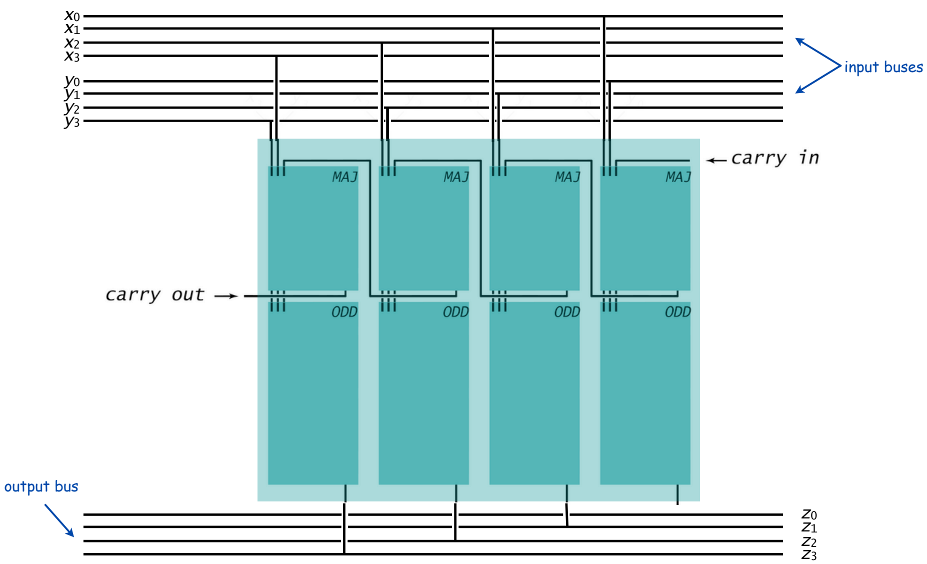

## Adder: Interface



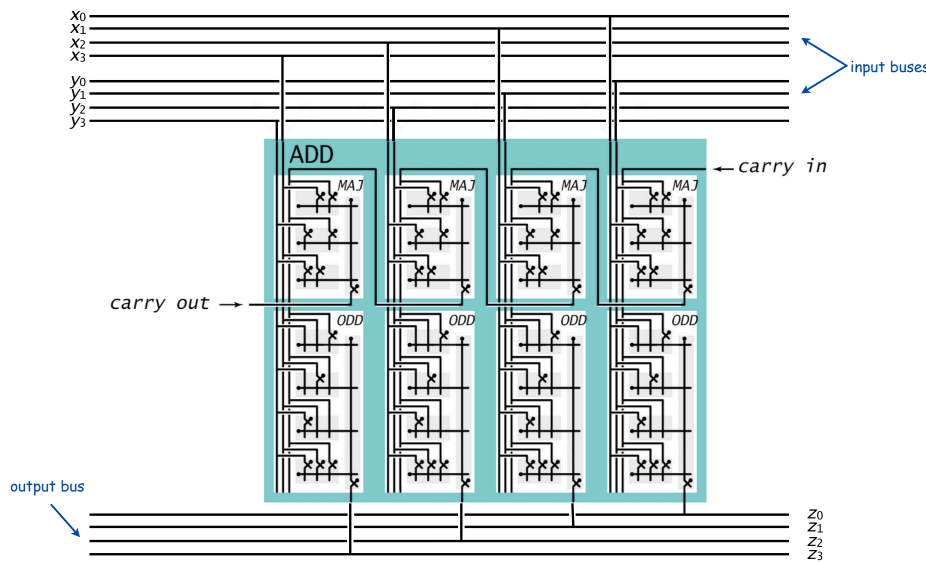

A bus is a group of wires that connect (carry data values) to other components.



## Adder: Component Level View

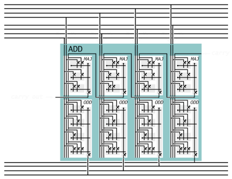



## Adder: Switch Level View

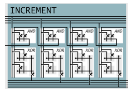


## Useful Combinational Circuits

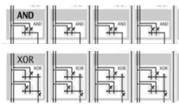
Adder



Incrementer (easy, add 0001)

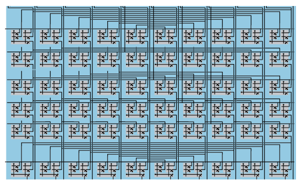



Bitwise AND, XOR (easy)




Decoder [next slide]

Shifter (clever, but we'll skip details)



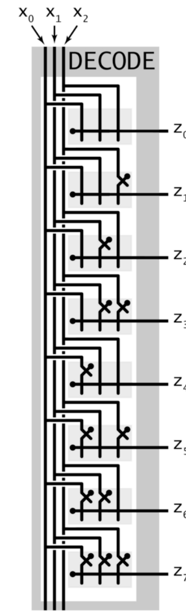
Multiplexer [next lecture]



## Decoder

Decoder. [n-bit]
- n address inputs, $2^n$ data outputs.
- Addressed output bit is 1; others are 0.
- Compact implementation of n Boolean functions

| $x_0$ | $x_1$ | $x_2$ | $z_0$ | $z_1$ | $z_2$ | $z_3$ | $z_4$ | $z_5$ | $z_6$ | $z_7$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |



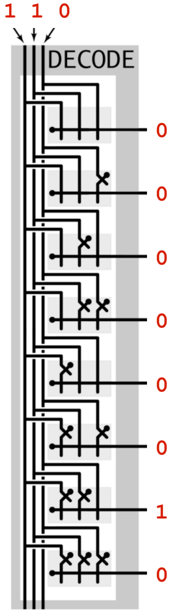

3-bit Decoder



## Decoder

Decoder. [n-bit]
- n address inputs, $2^n$ data outputs.
- Addressed output bit is 1; others are 0.
- Compact implementation of n Boolean functions

| $x_0$ | $x_1$ | $x_2$ | $z_0$ | $z_1$ | $z_2$ | $z_3$ | $z_4$ | $z_5$ | $z_6$ | $z_7$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |




3-bit Decoder

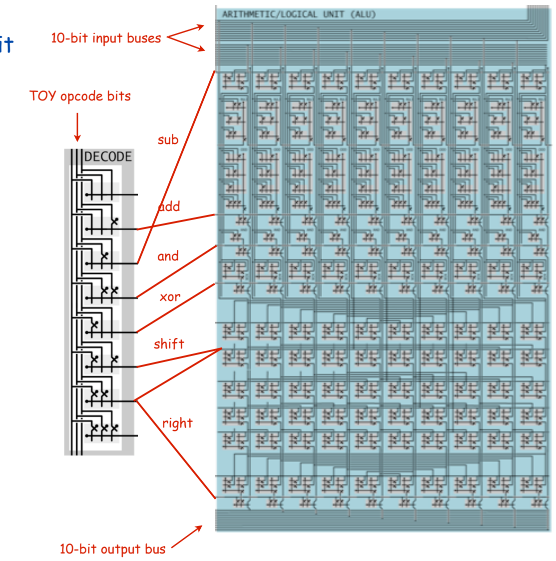


## Decoder application: Your computer's ALU!

ALU: Arithmetic and Logic Unit
- implements instructions
- input, output connects to registers via buses

Ex: TOY-Lite (10 bit words)
1: add
2: subtract
3: and
4: xor
5: shift left
6: shift right

Details:
- All circuits compute their result.
- Decoder lines AND all results.
- "one-hot" OR collects answer.



# Summary

Lessons for software design apply to hardware design!

- Interface describes behavior of circuit.
- Implementation gives details of how to build it.

Layers of abstraction apply with a vengeance!

- On/off.
- Controlled switch. [relay, transistor]
- Gates. [AND, OR, NOT]
- Boolean circuit. [MAJ, ODD]
- Adder.
- Shifter.
- Arithmetic logic unit.
- …
- TOY machine (stay tuned).
- Your computer.