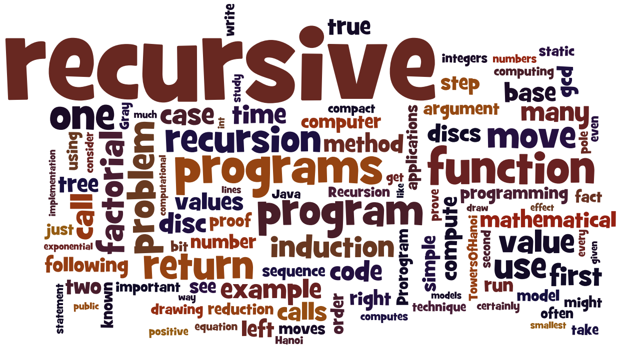
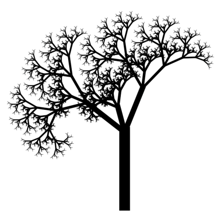


2.3 Recursion

Overview

What is recursion? When one function calls itself directly or indirectly.



Why learn recursion?

- New mode of thinking.
- Powerful programming paradigm.

Many computations are naturally self-referential.

- Binary search, mergesort, FFT, GCD.
- Linked data structures.
- A folder contains files and other folders.

Closely related to mathematical induction.

M. C. Escher, 1956

Greatest Common Divisor

Gcd. Find largest integer that evenly divides into p and q.

Ex. gcd(4032, 1272) = 24.

$$4032 = 2^6 \times 3^2 \times 7^1$$
$$1272 = 2^3 \times 3^1 \times 53^1$$
$$\text{gcd} = 2^3 \times 3^1 = 24$$

Applications.

- Simplify fractions: 1272/4032 = 53/168.
- RSA cryptosystem.

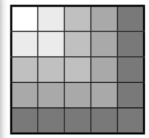
Mathematical Induction

Mathematical induction. Prove a statement involving an integer N by

- base case: Prove it for some specific N (usually 0 or 1).
- induction step: Assume it to be true for all positive integers less than N, use that fact to prove it for N.

Ex. Sum of the first N odd integers is N^2.

$$
\begin{aligned}
1 &= 1\\
1+3 &= 4\\
1+3+5 &= 9\\
1+3+5+7 &= 16\\
1+3+5+7+9 &= 25\\
\ldots &\ \ldots
\end{aligned}
$$



Base case: True for N = 1.

Induction step:

- Let T(N) be the sum of the first N odd integers: 1 + 3 + 5 + ... + (2N - 1).
- Assume that $T(N-1) = (N-1)^2$.
- $T(N) = T(N-1) + (2N - 1)$
  $= (N-1)^2 + (2N - 1)$
  $= N^2 - 2N + 1 + (2N - 1)$
  $= N^2$

Recursive Program

Recursive Program. Implement a function having integer arguments by

- base case: Implementing it for some specific values of the arguments.
- reduction step: Assume the function works for smaller values of its arguments and use it to implement it for the given values.

Ex. gcd(p, q).

Base case: gcd(p, 0) = p.

Reduction step: gcd(p, q) = gcd(p, p-q) if p-q > 0
= gcd(p, p-2q) if p-2q > 0
...
= gcd(p, p % q)

Greatest Common Divisor

GCD. Find largest integer that evenly divides into p and q.

Euclid's algorithm. [Euclid 300 BCE]

$$\gcd(p,q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q,\ p \ \%\ q) & \text{otherwise} \end{cases}$$

← base case
← reduction step, converges to base case

```
gcd(4032, 1272)   = gcd(1272, 216)
                  = gcd(216, 192)
                  = gcd(192, 24)
                  = gcd(24, 0)
                  = 24.
```

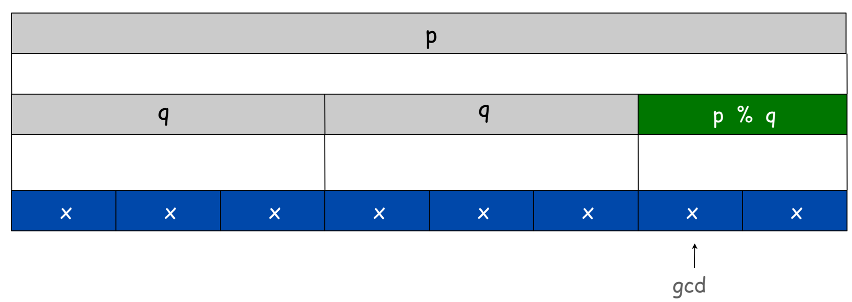
4032 = 3 × 1272 + 216

Euclid's Algorithm

GCD. Find largest integer d that evenly divides into p and q.

$$\gcd(p,q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q,\ p \ \%\ q) & \text{otherwise} \end{cases}$$

← base case
← reduction step, converges to base case



p = 8x
q = 3x

gcd(p, q) = gcd(3x, 2x) = x

Euclid's Algorithm

GCD. Find largest integer d that evenly divides into p and q.

$$\gcd(p,q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q,\, p \,\%\, q) & \text{otherwise} \end{cases}$$

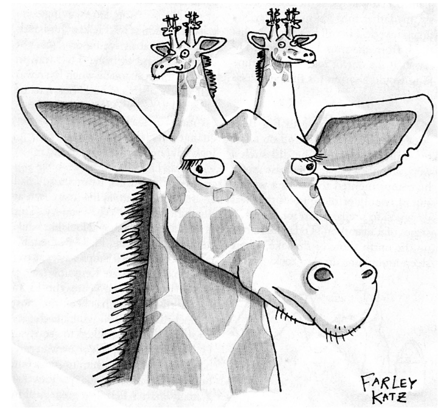
← base case

← reduction step, converges to base case

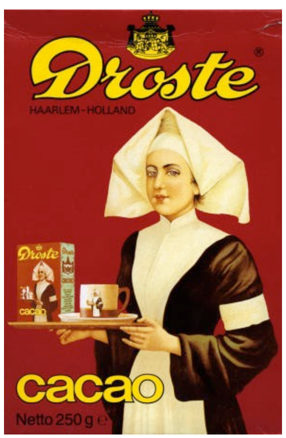
Recursive program

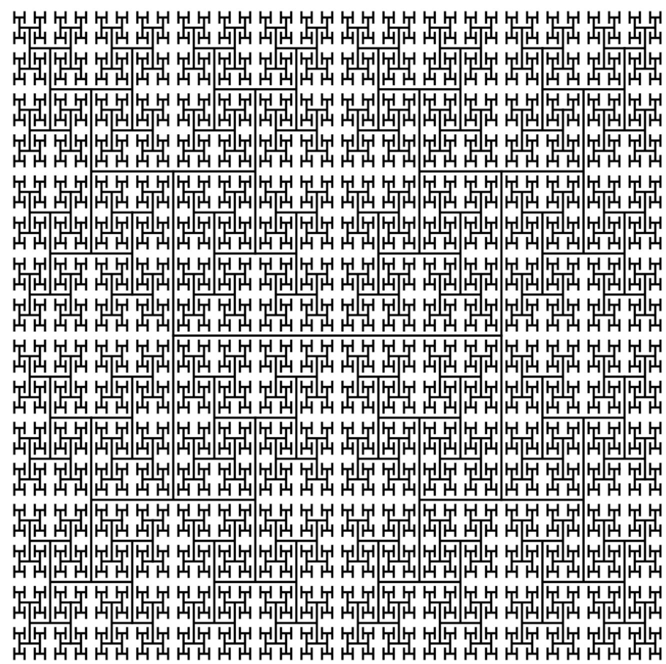
```
public static int gcd(int p, int q)
{
   if (q == 0) return p;            ← base case
   else return gcd(q, p % q);       ← reduction step
}
```


Recursive Graphics



New Yorker Magazine, August 11, 2008

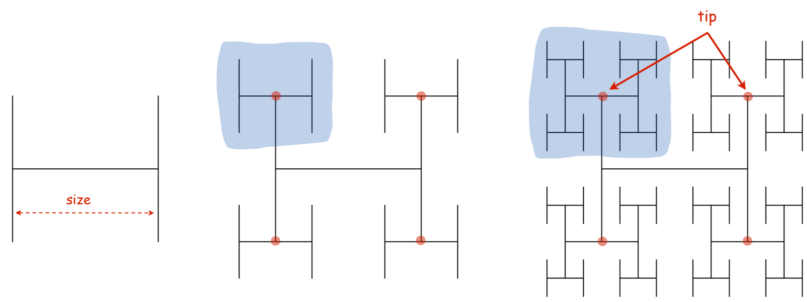




Htree

H-tree of order n.
- Draw an H.
- Recursively draw 4 H-trees of order n-1, one connected to each tip.

and half the size



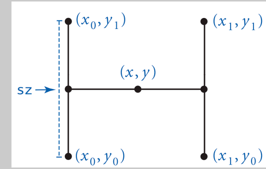
order 1   order 2   order 3

Htree in Java

```
public class Htree
{
   public static void draw(int n, double sz, double x, double y)
   {
      if (n == 0) return;
      double x0 = x - sz/2, x1 = x + sz/2;
      double y0 = y - sz/2, y1 = y + sz/2;

      StdDraw.line(x0,  y, x1,  y);
      StdDraw.line(x0, y0, x0, y1);
      StdDraw.line(x1, y0, x1, y1);

      draw(n-1, sz/2, x0, y0);
      draw(n-1, sz/2, x0, y1);
      draw(n-1, sz/2, x1, y0);
      draw(n-1, sz/2, x1, y1);
   }

   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      draw(n, .5, .5, .5);
   }
}
```

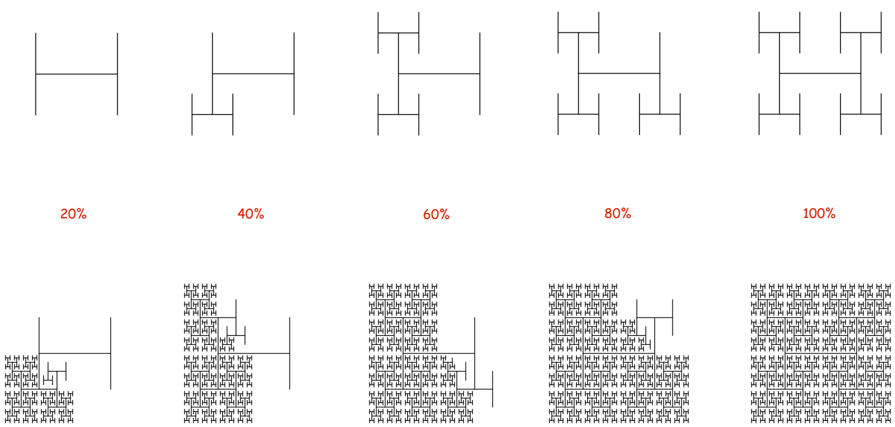
← draw the H, centered on (x, y)

← recursively draw 4 half-size Hs



Animated H-tree

Animated H-tree. Pause for 1 second after drawing each H.



Divide-and-Conquer

Divide-and-conquer paradigm.
- Break up problem into smaller subproblems of same structure.
- Solve subproblems recursively using same method.
- Combine results to produce solution to original problem.

Divide et impera. Veni, vidi, vici. - Julius Caesar

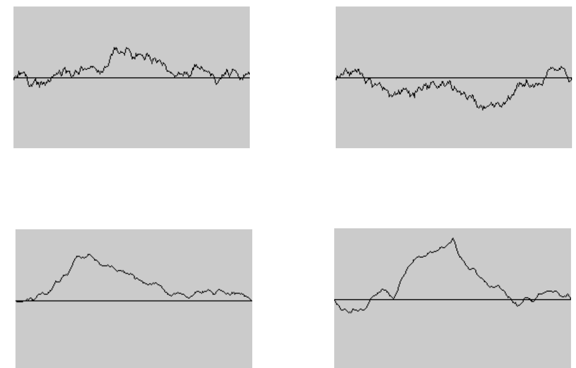
Many important problems succumb to divide-and-conquer.
- FFT for signal processing.
- Parsers for programming languages.
- Multigrid methods for solving PDEs.
- Quicksort and mergesort for sorting.
- Hilbert curve for domain decomposition.
- Quad-tree for efficient N-body simulation.
- Midpoint displacement method for fractional Brownian motion.

Application: Fractional Brownian Motion

Fractional Brownian Motion

Physical process which models many natural and artificial phenomenon.

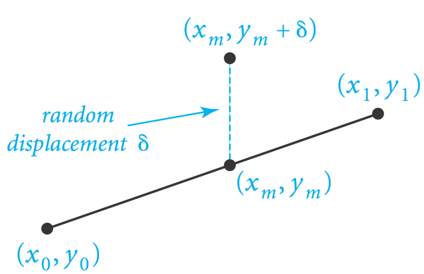
- Price of stocks.
- Dispersion of ink flowing in water.
- Rugged shapes of mountains and clouds.
- Fractal landscapes and textures for computer graphics.



Simulating Brownian Motion

Midpoint displacement method.

- Maintain an interval with endpoints (x_0, y_0) and (x_1, y_1).
- Divide the interval in half.
- Choose δ at random from Gaussian distribution.
- Set $x_m = (x_0 + x_1)/2$ and $y_m = (y_0 + y_1)/2 + \delta$.
- Recur on the left and right intervals.



Simulating Brownian Motion: Java Implementation

Midpoint displacement method.

- Maintain an interval with endpoints (x_0, y_0) and (x_1, y_1).
- Choose δ at random from Gaussian distribution.
- Divide the interval in half: Set $x_m = (x_0 + x_1)/2$ and $y_m = (y_0 + y_1)/2 + \delta$.
- Recur on the left and right intervals.

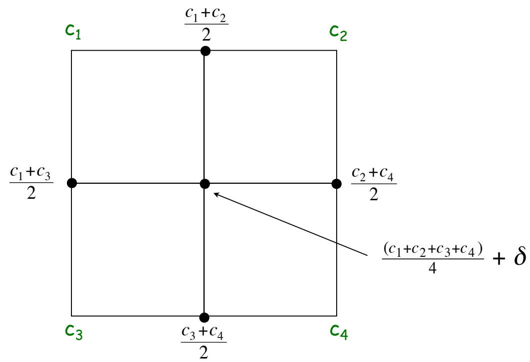
```
public static void curve(double x0, double y0,
                         double x1, double y1, double var)
{
   if (x1 - x0 < 0.01)
   {
      StdDraw.line(x0, y0, x1, y1);
      return;
   }
   double xm = (x0 + x1) / 2;
   double ym = (y0 + y1) / 2;
   ym += StdRandom.gaussian(0, Math.sqrt(var));
   curve(x0, y0, xm, ym, var/2);
   curve(xm, ym, x1, y1, var/2);
}
```

variance halves at each level;
change factor to get different shapes

Plasma Cloud

Plasma cloud centered at (x, y) of size s.

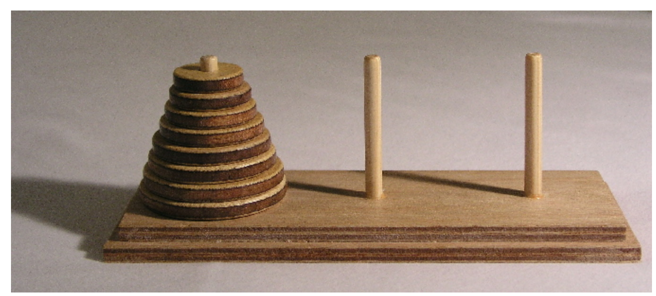
- Each corner labeled with some grayscale value.
- Divide square into four quadrants.
- The grayscale of each new corner is the average of others.
  - center: average of the four corners + random displacement
  - others: average of two original corners
- Recur on the four quadrants.



Plasma Cloud

Brownian Landscape

Towers of Hanoi

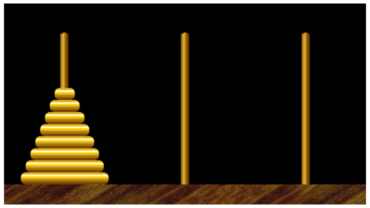


http://en.wikipedia.org/wiki/Image:Hanoiklein.jpg

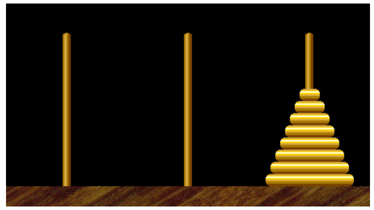
Towers of Hanoi

Move all the discs from the leftmost peg to the rightmost one.

- Only one disc may be moved at a time.
- A disc can be placed either on empty peg or on top of a larger disc.



start

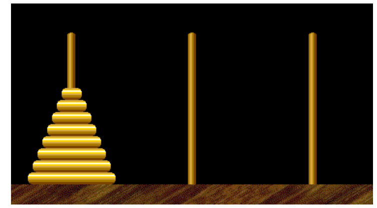


finish

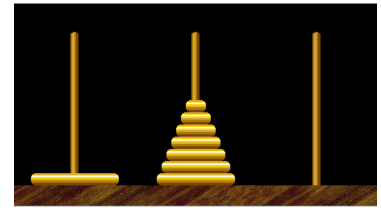
Towers of Hanoi demo

Edouard Lucas (1883)

Towers of Hanoi: Recursive Solution

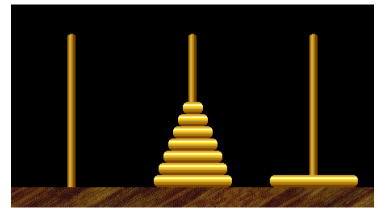


Move n-1 smallest discs right.

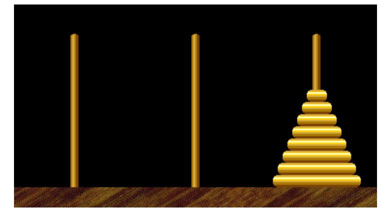


Move largest disc left.

cyclic wrap-around



Move n-1 smallest discs right.



Towers of Hanoi Legend

Q. Is world going to end (according to legend)?

- 64 golden discs on 3 diamond pegs.
- World ends when certain group of monks accomplish task.

Q. Will computer algorithms help?

Towers of Hanoi: Recursive Solution

```
public class TowersOfHanoi
{
   public static void moves(int n, boolean left)
   {
      if (n == 0) return;
      moves(n-1, !left);
      if (left) System.out.println(n + " left");
      else      System.out.println(n + " right");
      moves(n-1, !left);
   }

   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      moves(N, true);
   }

}
```

`moves(n, true)` : move discs 1 to n one pole to the left
`moves(n, false)`: move discs 1 to n one pole to the right

smallest disc

Towers of Hanoi: Recursive Solution

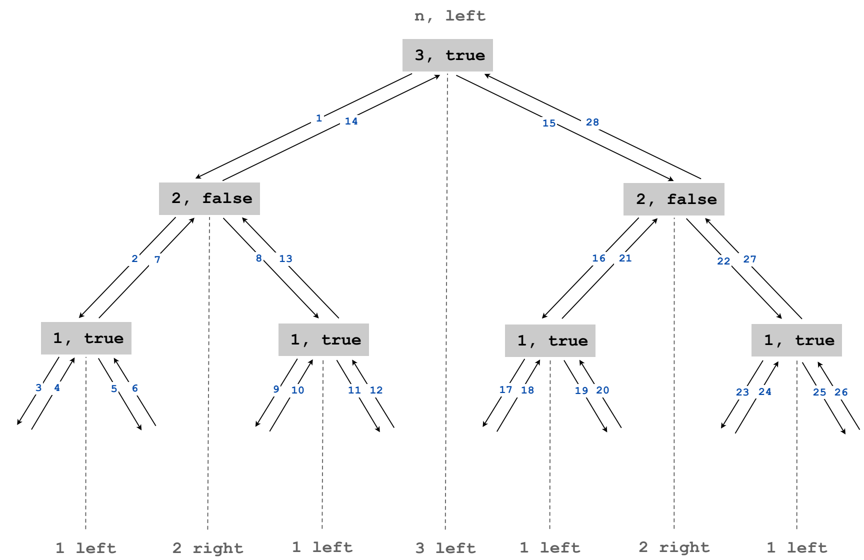
```
% java TowersOfHanoi 3
1 left
2 right
1 left
3 left
1 left
2 right
1 left
```

```
% java TowersOfHanoi 4
1 right
2 left
1 right
3 right
1 right
2 left
1 right
4 left
1 right
2 left
1 right
3 right
1 right
2 left
1 right
```

every other move is smallest disc

subdivisions of ruler

Towers of Hanoi: Recursion Tree



Towers of Hanoi: Properties of Solution

Remarkable properties of recursive solution.

- Takes $2^n - 1$ moves to solve n disc problem.
- Sequence of discs is same as subdivisions of ruler.
- Every other move involves smallest disc.

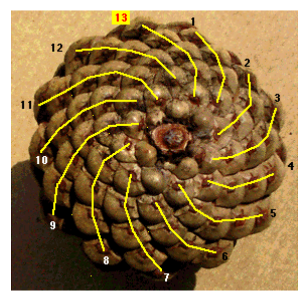
Recursive algorithm yields non-recursive solution!

- Alternate between two moves: to left if n is odd
  - move smallest disc to right if n is even
  - make only legal move not involving smallest disc

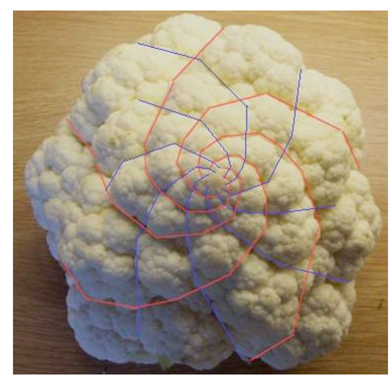
Recursive algorithm may reveal fate of world.

- Takes 585 billion years for n = 64 (at rate of 1 disc per second).
- Reassuring fact: any solution takes at least this long!

Fibonacci Numbers



pinecone

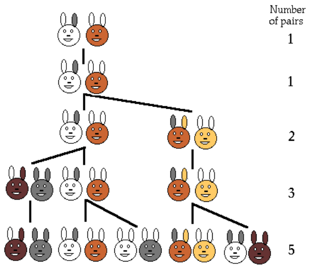


cauliflower

Fibonacci Numbers

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, …

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$



Fibonacci rabbits

L. P. Fibonacci
(1170 - 1250)

A Possible Pitfall With Recursion

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, …

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

FYI (classical math):

$$F(n) = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}} = \left\lfloor \phi^n / \sqrt{5} \right\rceil$$

ϕ = golden ratio $\approx$ 1.618

Ex: F(50) $\approx 1.2 \times 10^{10}$

A natural for recursion?

```
public static long F(int n)
{
   if (n == 0) return 0;
   if (n == 1) return 1;
   return F(n-1) + F(n-2);
}
```


TEQ on Recursion 1.1 (difficult but important)

Is this an efficient way to compute F(50)?

```
public static long F(int n)
{
   if (n == 0) return 0;
   if (n == 1) return 1;
   return F(n-1) + F(n-2);
}
```


TEQ on Recursion 1.2 (easy and also important)

Is this an efficient way to compute F(50)?

```
long[] F = new long[51];
F[0] = 0; F[1] = 1;
if (n == 1) return 1;
for (int i = 2; i <= 50; i++)
   F[i] = F[i-1] + F[i-2];
```

Summary

How to write simple recursive programs?

- Base case, reduction step.
- Trace the execution of a recursive program.
- Use pictures.

Why learn recursion?

Towers of Hanoi by W. A. Schloss.

- New mode of thinking.
- Powerful programming tool.

Divide-and-conquer. Elegant solution to many important problems.

Exponential time.

- Easy to specify recursive program that takes exponential time.
- Don't do it unless you plan to (and are working on a small problem).