COS 597C: How to Solve it **Fall 2009**

Assignment #2

Due: Thursday, October 1 *Sanjeev Arora*

Please be succinct, and give a proof sketch rather than gory details.

1. What is the expected number of triangles in $G(n, p)$ when you plant a random clique of size k in it? Try to prove concentration of this quantity.

2. Let the *Frobenius* norm of a real symmetric $n \times n$ matrix M, denote $|M|_F^2$, be defined as the sum of squares of its entries. Let its *spectral* norm, denoted $|M|$, be the magnitude of the largest eigenvalue. Show that if M has rank k then $|M|_F^2 \leq k|M|^2$.

3. Suppose M is an $n \times n$ symmetric matrix and $u_1, u_2, \ldots, u_k$ are the eigenvectors corresponding to its k largest eigenvalues (in absolute value). Let $\hat{M}$ be the matrix obtained by replacing each column of M by its projection in the k-dimensional subspace spanned by $u_1, u_2, \ldots, u_k$. Note that $\hat{M}$ has rank at most k. Show that for every matrix X of rank k, $|M - \hat{M}| \leq |M - X|$.

4. The *trace* of a matrix, denoted $Tr(M)$, is the sum of its diagonal entries. Show that for every real symmetric matrix, $\lim_{k\to\infty} Tr(M^{2k})^{1/2k}$ exists and is equal to $|M|$. (Here the limit is taken over integral k.)