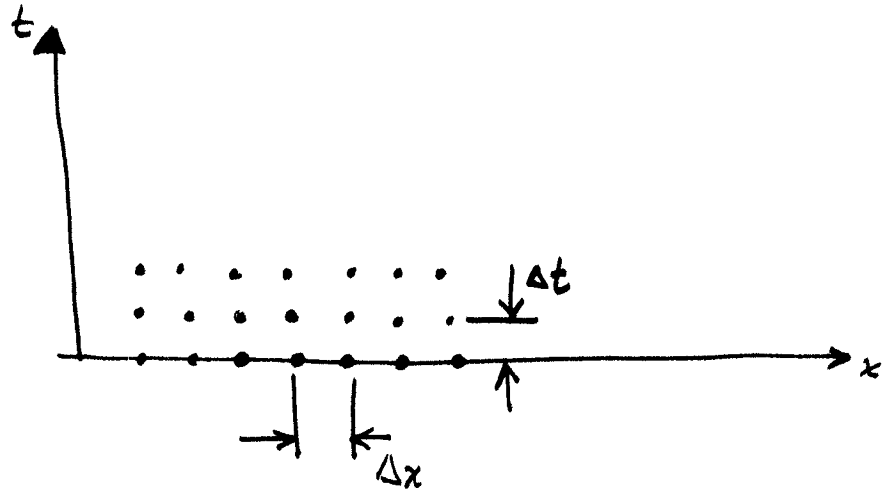
Numerical Solution of Wave Equation

[Ames 92]
[Smith 85]

Grid



$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Use central difference approximation to $\frac{\partial^2}{\partial^2}$

$$y(x,t) = y(m\Delta x, n\Delta t) = y_{mn}$$

$$\frac{y_{m,n+1} - 2y_{mn} + y_{m,n-1}}{(\Delta t)^2} = c^2 \frac{y_{m+1,n} - 2y_{mn} + y_{m-1,n}}{(\Delta x)^2} + O(\Delta t)^2 + O(\Delta x)^2$$

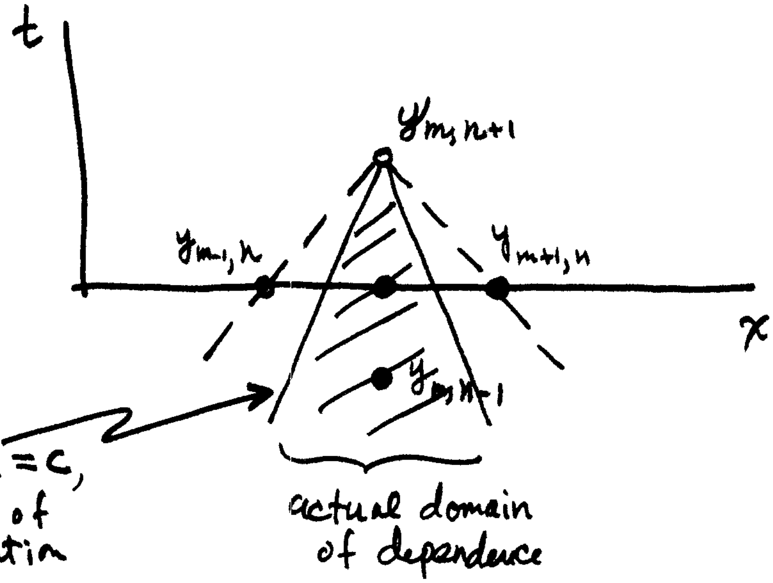
future value

$$y_{m,n+1} = c^2\left(\frac{\Delta t}{\Delta x}\right)^2 \left[y_{m+1,n} - 2y_{mn} + y_{m-1,n}\right] + 2y_{mn} - y_{m,n-1}$$

Let $s^2 = c^2\left(\frac{\Delta t}{\Delta x}\right)^2$, dimensionless parameter

$$\boxed{y_{m,n+1} = 2(1-s^2)y_{mn} + s^2 y_{m+1,n} + s^2 y_{m-1,n} - y_{m,n-1}}$$

Intuitive stability analysis:



$\frac{dx}{dt}$ = slope = c,
Velocity of propagation

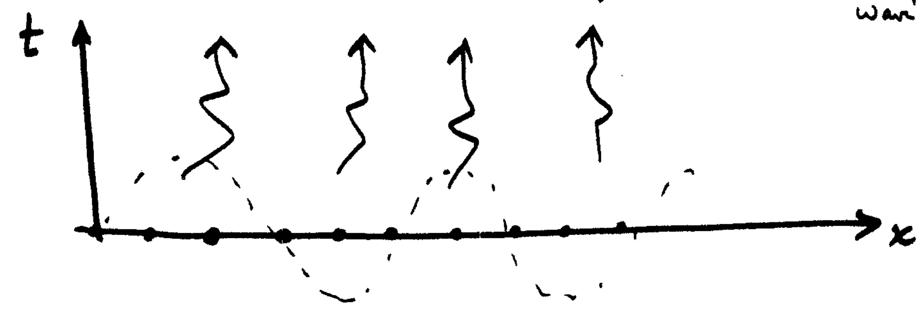
For the numerical method to capture the correct behavior, the actual domain of dependence must lie inside the numerical domain.

∴ it is necessary that $\frac{\Delta x}{\Delta t} \geq c$

$$\boxed{\frac{c \cdot \Delta t}{\Delta x} = s \leq 1}$$

Von Neumann Stability Analysis

[Smith85, Ex. 2.12, p.70]
for explicit scheme for wave eqn.



idea: • assume a sinusoidal component at $t=0$

• study propagation in t

Use complex exponential representation

$$y_{m,0} = e^{j\lambda m \Delta x}$$

must be stable for <u>any</u> $\underline{\lambda}$.

Guess propagated waveform is of the form

$$y_{m,n} = e^{j\lambda m \Delta x} \cdot \xi^n$$

(can be justified by separation of variables)

ξ is unknown, but we want $|\xi| \le 1$

"amplification"

this is actually only a <u>necessary</u> condition.
(But it's often sufficient.)

Substitute this y_{mn} into difference eqn:

$$y_{m,n+1} = 2(1-s^2)y_{mn} + s^2 y_{m+1,n} + s^2 y_{m-1,n} - y_{m,n-1}$$

$$\begin{cases} y_{mn} = e^{j\lambda m\Delta x}\xi^n = Y \\ y_{m,n+1} = \xi\cdot Y \\ y_{m\pm1,n} = e^{\pm j\lambda\Delta x}\cdot Y \\ y_{m,n-1} = \xi^{-1}\cdot Y \end{cases}$$

$$\xi\cdot\cancel{Y} = 2(1-s^2)\cancel{Y} + s^2(e^{j\lambda\Delta x}+e^{-j\lambda\Delta x})\cdot\cancel{Y} - \xi^{-1}\cancel{Y}$$

$$\xi = 2(1-s^2) + 2s^2\cos(\lambda\Delta x) - \xi^{-1}$$

$$\xi = 2 + 2s^2(\underbrace{\cos\lambda\Delta x - 1}_{2\sin^2\frac{\lambda\Delta x}{2}}) - \xi^{-1}$$

$$\xi^2 - 2\left[1 - 2s^2\sin^2\frac{\lambda\Delta x}{2}\right]\xi + 1 = 0$$

Product of roots $\xi_1\cdot\xi_2 = 1$

not both 1, so roots should be <u>complex</u>:

$$b^2 - 4ac = 4\left(1-2s^2\sin^2\frac{\lambda\Delta x}{2}\right)^2 - 4 \le 0$$

$$\Rightarrow \left(1-2s^2\sin^2\frac{\lambda\Delta x}{2}\right)^2 \le 1$$

$$\Rightarrow \pm\left(1-2s^2\sin^2\frac{\lambda\Delta x}{2}\right) \le 1$$ (+ condition is trivially true)

$$\Rightarrow s^2\sin^2\frac{\lambda\Delta x}{2} \le 1$$

$$\Rightarrow s^2 \le 1/\sin^2\left(\frac{\lambda\Delta x}{2}\right)$$ <u>all</u> λ !

$$\Rightarrow \boxed{s^2 \le 1} \Rightarrow \boxed{|s| \le 1}$$ (also sufficient)

this wave equation approximation is <u>consistent</u>,

$$\frac{y_{m,n+1}-2y_{m,n}+y_{m,n-1}}{k^2} = c^2\,\frac{y_{m+1,n}-2y_{mn}+y_{m-1,n}}{h^2} + \underbrace{O(k^2)+O(h^2)}_{\to\, 0 \text{ as } k,h\to 0}$$

Local truncation error

$\therefore$ when $|\alpha|\le 1$ this <u>stable</u> & <u>convergent</u> $\to$ (by Lax's thm.)

<u>Another example</u>: the diffusion eqn. approximation (p. 4.1.11)

$$\frac{y_{i,j+1}-y_{i,j}}{k} = \frac{y_{i-1,j}-2y_{ij}+y_{i+1,j}}{h^2} + \underbrace{O(k)+O(h^2)}_{\to 0}$$

$\Rightarrow$ <u>consistent</u>

[Smith85, Ex. 2.11, p.69]

Von Neumann's method:

$$y_{i,j+1} = r\,y_{i-1,j} + (1-2r)\,y_{ij} + r\,y_{i+1,j}$$

$$\xi\cdot\cancel{Y} = r\left(e^{j\lambda\Delta x}+e^{-j\lambda\Delta x}\right)\cdot\cancel{Y} + (1-2r)\cdot\cancel{Y}$$

$$\xi = 2r\left(\cos(\lambda\Delta x)-1\right)+1$$

$$|\xi| = \left|1-4r\sin^2\frac{\lambda\Delta x}{2}\right| \le 1$$

$$\Rightarrow\quad r \le \frac{1}{2\sin^2\frac{\lambda\Delta x}{2}} \qquad \text{all } \lambda$$

$$\Rightarrow\quad \boxed{r\le 1/2} \qquad \text{stable \& convergent here}$$

Example of Inconsistency [Smith 85, Ex 2.7, p. 42] [Ames 92, p. 71-2, 64]

Richardson in 1910 suggested using central difference approximation for $\partial y/\partial t$ to approximate diffusion eqn.

$$\frac{\partial y}{\partial t} = \frac{y_{i,j+1} - y_{i,j-1}}{2k} + O(k^2) = \frac{y_{i-1,j} - 2y_{ij} + y_{i+1,j}}{h^2} + O(h^2)$$

this turns out to be <u>always</u> <u>unstable</u>!

<u>DuFort & Frankel</u> in 1953 suggested replacing y_{ij} on RHS using

$$\frac{1}{2}\left(y_{i,j+1} + y_{i,j+1}\right) \approx y_{ij}$$

this results in a difference eqn. that is <u>always stable</u>.

Actually, this replacement introduces the term

$$\frac{2y_{ij} - (y_{i,j+1} + y_{i,j-1})}{h^2} = -\left(\frac{k^2}{h^2}\right)\frac{\partial^2 y}{\partial t^2} + O\left(\frac{k^4}{h^2}\right)$$

because

$$\frac{2y_{ij} - (y_{i,j+1} + y_{i,j-1})}{k^2} = -\frac{\partial^2 y}{\partial t^2} + O(k^2)$$

— p. 4.1.9

If $\frac{k}{h} = r = \text{const.}$, and $k \to 0$, this approximation is consistent with the the partial differential eqn.

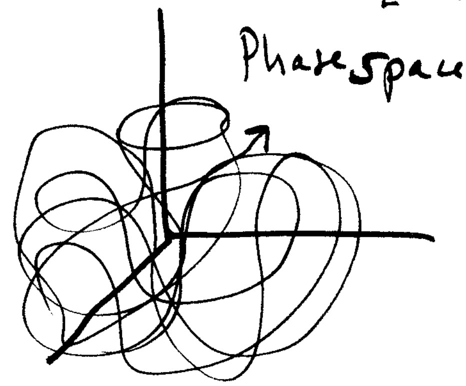
$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} - r^2 \frac{\partial^2 y}{\partial t^2},$$

— <u>not</u> the original diffusion eqn.!

The Fermi-Pasta-Ulam Experiment (1955) [FPU65] [GT96]

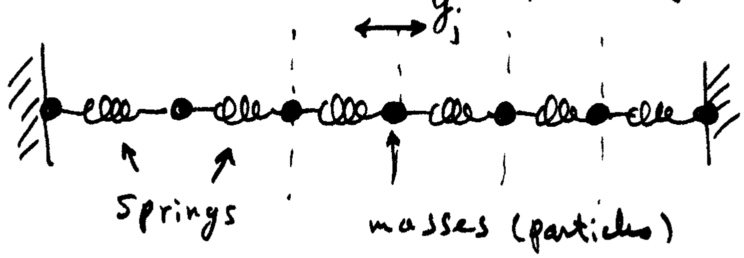
the expectation

"Ergodic" → all regions traversed equally often if system is nonlinear.



this idea shattered... lead to solitons.

their model was coupled oscillators y_j = position of jth particle



$$\text{Force on particle } j = m \cdot a = m \frac{\partial^2 y_j}{\partial t^2} = -K(y_j - y_{j+1}) - K(y_j - y_{j-1})$$

← restoring forces, K is spring constant

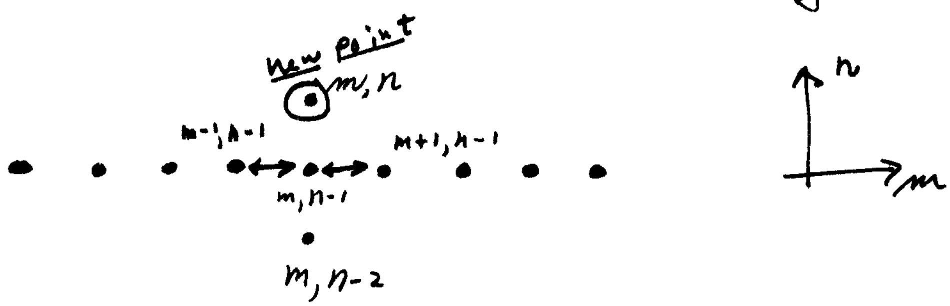
$$\frac{\partial^2 y_j}{\partial t^2} = -\frac{K}{m}\left[2y_j - y_{j-1} - y_{j+1}\right]$$

$$\frac{\partial^2 y_j}{\partial t^2} = \frac{K}{m}\underbrace{\left[y_{j+1} + y_{j-1} - 2y_j\right]}_{\approx \frac{\partial^2 y_j}{\partial x^2}}$$

approximates wave eqtn.

to incorporate nonlinear coupling terms, FPU add forces dependant on square or cube of inter-particle distances.

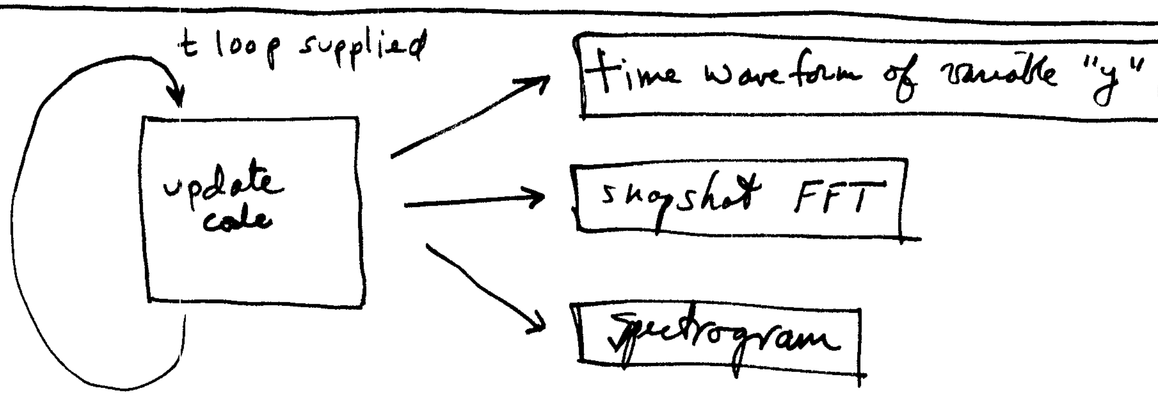
We'll do the same in our solution of wave equation:



$$y_{m,n} = \underbrace{\sim\sim\sim}_{\text{linear part}} + \alpha \underbrace{\overline{\qquad\qquad}}_{\text{nonlinear part}}$$

Start with all energy in mode 0: 

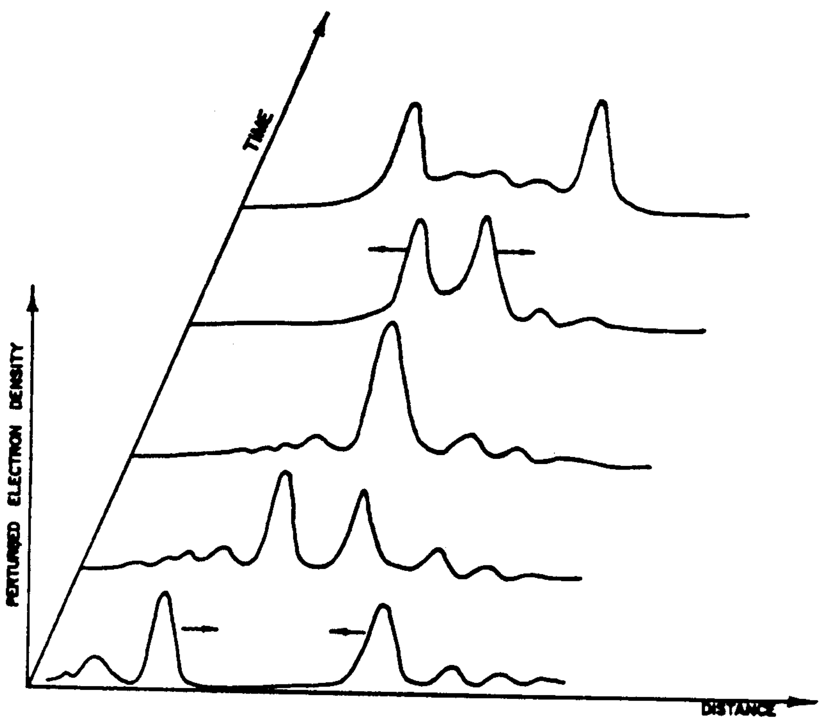
EIN



Solitons

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon . . . .

J. Scott-Russell, 1844
in A.C. Scott et al. [SCM73]



Solitons

Fig. 4. Nondestructive collision of ion-acoustic plasma pulses.

H. Izeki et al. in [SCM73]

Plasmas
elastic rods
pressure waves in liquid-gas bubble mixtures
phonon packets in nonlinear crystals
shallow water
nonlinear optical fibers
Josephson junction
electrical lattices ... etc.

History of Soliton Experiment & theory

4.3.10

1834 Scott-Russell on horseback, in canal

1895 Korteweg & deVries write differential equation for shallow water (KdV eqn.)

1953 Seeger, Donth, & Kochendörfer
1962 Perring & Skyrme
observed nondestructive collisions

FPU 1955

1965 Zabusky & Kruskal Solitons in plasma

1973 Ablowitz, Kaup, Newell, Segur (analytical) inverse scattering method

⋮

leads to current explosion of work in nonlinear systems.

Linear systems

- Laplace's Eqn.
- Poisson's Eqn.
- Diffusion Eqn.
- wave Eqn.
- Schrödinger Eqn

⋮

main feature is SUPERPOSITION of Solutions

NonLinear Systems

- Logistic, e.g.
- KdV
- many nonlinear versions of Schrödinger Eqn.
- many saturating systems
- more realistic versions of oversimplified physical systems

→ Solitons

→ chaos

→ much more complex & interesting than Linear eqns.

Soliton on the Scott Russell Aqueduct on the Union Canal near Heriot-Watt University, 12 July 1995.

Full Size Version

Solitons Home Page

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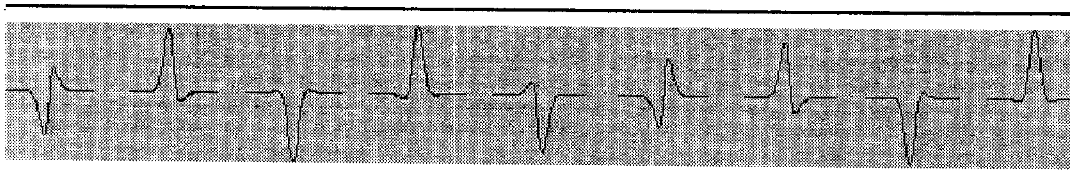

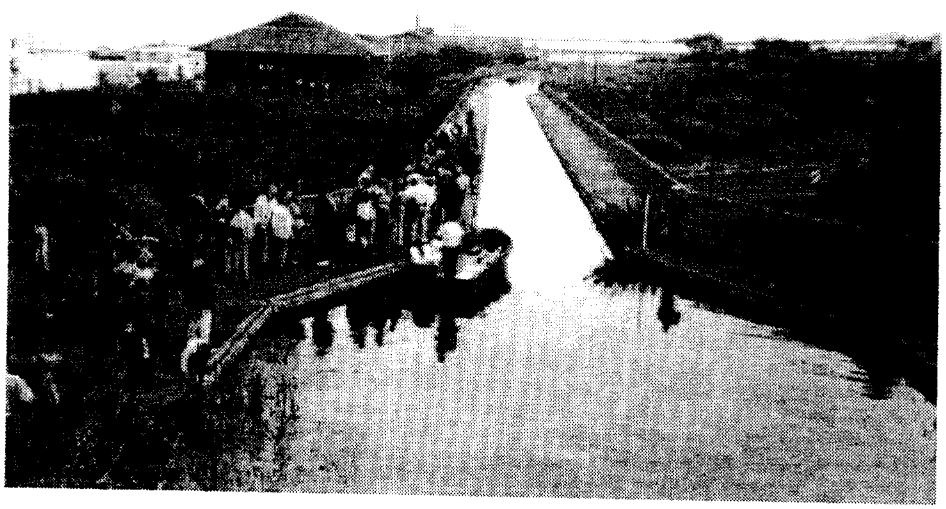

The Scott Russell Aqueduct on the Union Canal near Heriot-Watt University, 12 July 1995.

For the technically minded, the aqueduct is 89.3 m long, 4.13m wide, and 1.52m deep.