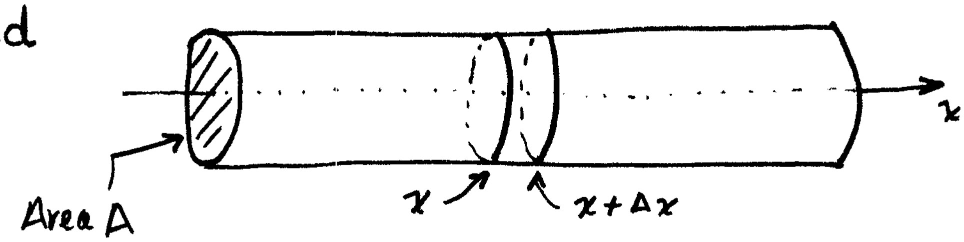
Where do PDE's come from?

Heat Equation (Diffusion)

[Powers79]
[EK88]
[SR66]

Rod



$\begin{cases} q = \text{rate of heat flow per unit area} \\ u = \text{temperature} \end{cases}$

$$\underbrace{\cancel{A}\, q(x,t)}_{\text{entering heat}} = \underbrace{\cancel{A}\, q(x+\Delta x, t)}_{\text{leaving heat}} + \text{rate of heat storage}$$

~ rate of change of temperature u

$$\cancel{A}\, \rho c\, \Delta x \frac{\partial u}{\partial t}$$

ρ : density

c : heat capacity per unit mass

$$\underbrace{\frac{q(x+\Delta x,t) - q(x,t)}{\Delta x}}_{\text{limit as } \Delta x \to 0} = \underbrace{-\rho c \frac{\partial u}{\partial t}}_{\text{independent of } \Delta x}$$

$$\frac{\partial q}{\partial x} = -\rho c \frac{\partial u}{\partial t}$$

rate of heat flow

temperature gradient

Fourier's LAW: "heat flows downhill," or $q = -K \dfrac{\partial u}{\partial x}$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{\rho c}{K} \frac{\partial u}{\partial t}}$$ Heat Eqn.

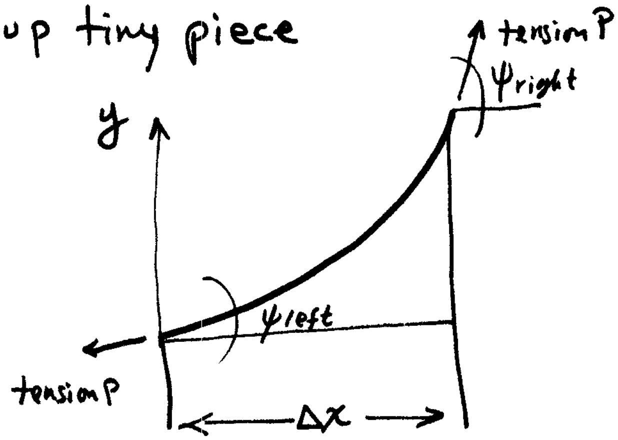
Wave Equation

Consider stretched string between two fixed points:

vibrates

[Steiglitz 96]
[Sokolnikoff 66]

blow up tiny piece



net vertical component of force on segment $= P \sin \psi_{right} - P \sin \psi_{left} = \underbrace{ma}_{\text{Newton's law}} = \underbrace{\rho \Delta x}_{\text{mass}} \dfrac{\partial^2 y}{\partial t^2}$

ρ: density

$$\frac{\partial^2 y}{\partial t^2} = \frac{P}{\rho} \left[\frac{\sin \psi_{right} - \sin \psi_{left}}{\Delta x} \right]$$

in reality, ψ is <u>very small</u>, so

$$\sin \psi \approx \tan \psi \approx \frac{\partial y}{\partial x}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{P}{\rho} \left[\frac{\left.\frac{\partial y}{\partial x}\right|_{right} - \left.\frac{\partial y}{\partial x}\right|_{left}}{\Delta x} \right]$$

limit as $\Delta x \to 0$

$$\boxed{\frac{\partial^2 y}{\partial t^2} = \underbrace{\frac{P}{\rho}}_{} \frac{\partial^2 y}{\partial x^2}}$$

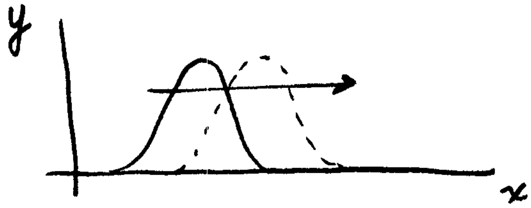
Wave Eqn.

usually denoted c^2, dimensions of velocity^2

we'll concentrate on wave eqn.

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}}$$

Guess a bump moving to the right on the string



this is of the form $f(t - x/c)$

speed is c: increase t: $t + \Delta t$ $\quad \left(t + \Delta t - \frac{x + c\Delta t}{c}\right) = t - x/c$ ✓

increase x: $x = x + c\Delta t$

$\rightarrow \Delta x / \Delta t = c$

Check Eqn: $\frac{\partial}{\partial t^2} = f''(t - x/c)$

$c^2 \frac{\partial^2}{\partial x^2} = \cancel{c^2} \frac{1}{\cancel{c^2}} f''(x - x/c)$ ✓

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this is true no matter what the shape $f(\cdot)$

Same argument works for bump moving <u>left</u>

$g(t + x/c)$

If any two solns. are added, the sum also satisfies wave eqn. So we always have as a soln.

$$\underbrace{f(t - x/c)}_{\text{right-moving wave}} + \underbrace{g(t + x/c)}_{\text{left-moving wave}}$$

general form

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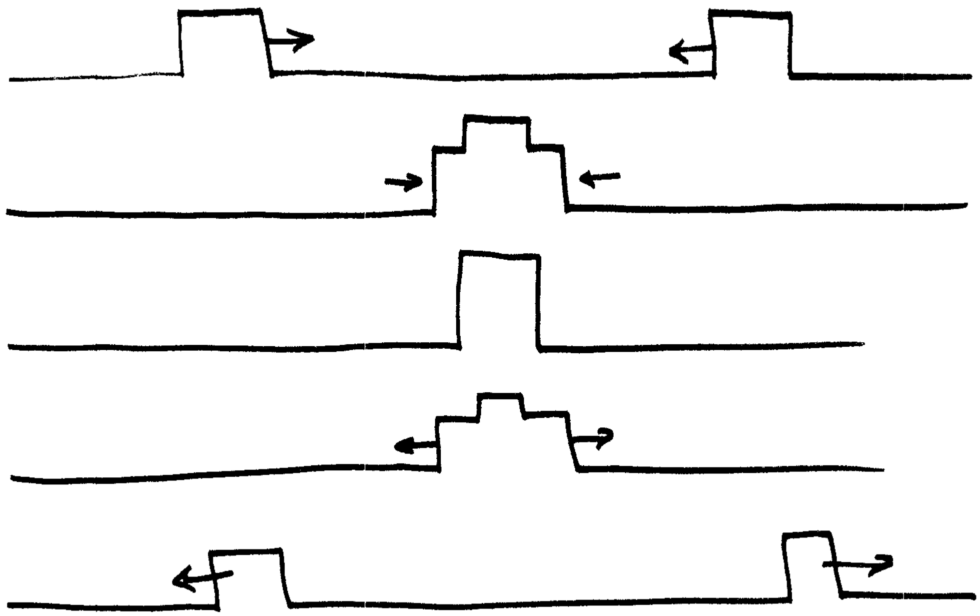
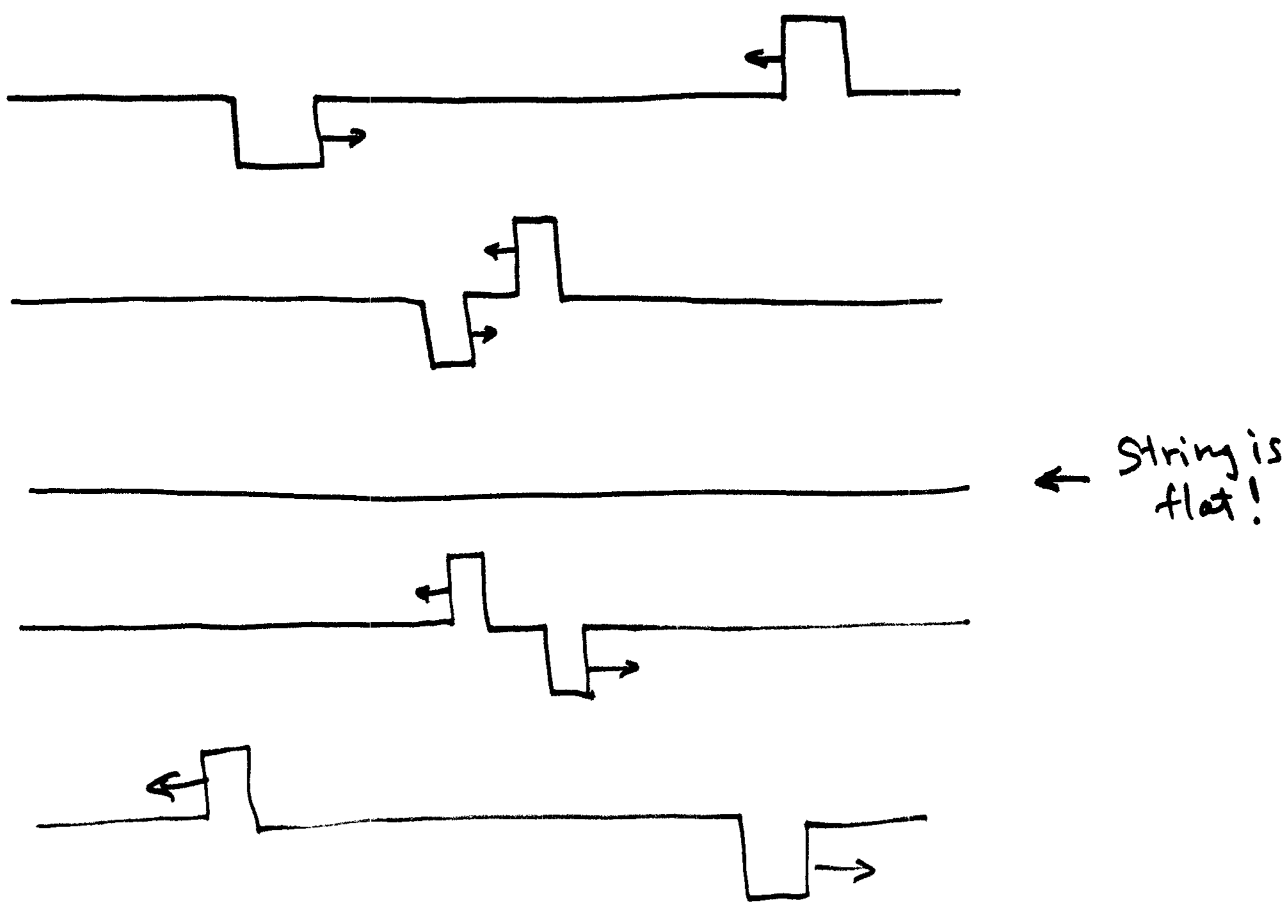
what causes periodic vibrations, like guitar string?

EXAMPLES:





Boundary Conditions String is tied down at ends, say:

Impose the condition that deflection at $x=0$ is 0:

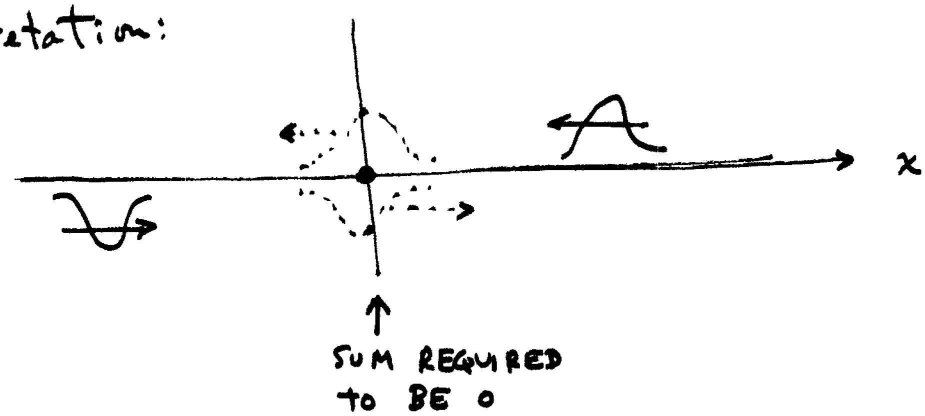
$$y(0,t) = f(t) + g(t) = 0, \quad \text{all } t$$

$$\Rightarrow \quad f(t) = -g(t)$$

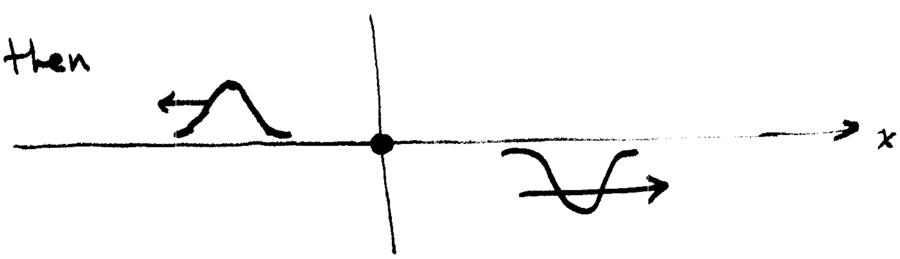
So general solution becomes

$$y(x,t) = f(t - x/c) - f(t + x/c)$$

Interpretation:



And then



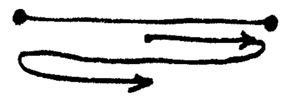
∴ wave shape is <u>inverted</u> on reflection from fixed end.

Second fixed point:

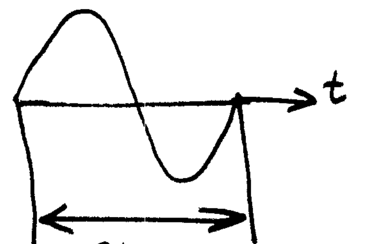
$$y(L,t) = f(t - L/c) - f(t + L/c) = 0$$

In other words, $f(t) = f(t + 2L/c)$ <u>periodic</u> in t

what is $\frac{2L}{c}$? Round<u>trip</u> time



We're going to look for solutions with this period



$T_0 = \frac{2L}{c}$ sec

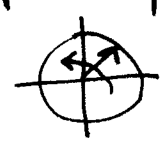
$f_0 = \frac{1}{T_0} = \frac{c}{2L}$ Hz (cycles/sec)

$\omega_0 = 2\pi f_0 = \frac{\pi c}{L}$ radians/sec

Also, instead of $\sin(\omega_0 t)$ & $\cos(\omega_0 t)$, we'll use

$$\boxed{e^{j\omega_0 t} = \cos(\omega_0 t) + j \sin(\omega_0 t)}$$

complex exponential



this form greatly simplifies algebra, and will be useful later for (1) stability analysis — and still later for Fourier analysis.

We guess a solution of the form

$$y(x,t) = \underbrace{e^{j\omega_0 t}}_{\text{periodic factor}} Y(x)$$

Plug into wave equation,

$$\frac{\partial^2 y}{\partial t^2} = -\omega_0^2 e^{j\omega_0 t} Y(x)$$

$$c^2 \frac{\partial^2 y}{\partial x^2} = c^2 e^{j\omega_0 t} Y''(x)$$

must be equal

$$Y''(x) = -\frac{\omega_0^2}{c^2} Y(x) = -\frac{\pi^2 \cancel{c^2}}{L^2 \cancel{c^2}} Y(x)$$

(this is an ODE)

Solution:

$$Y(x) = \sin(\pi x/L + \phi)$$

We have yet to determine ϕ, so use boundary conditions

$$Y(x) = 0 \quad \text{at} \quad x = 0 \text{ and } L.$$

yields

$$\sin\phi = \sin(\pi + \phi) = 0$$

$$\Rightarrow \phi = 0, \pm\pi, \pm 2\pi, \cdots$$

Doesn't matter, use $\phi = 0$, so solution is

$$\boxed{y(x,t) = e^{j\omega_0 t} \sin(\pi x/L)}$$

(If you want, take real part, since this is complex.)

<u>But</u>, we could have also tried

$$e^{j(2\omega_0)t}\, Y(x)$$

$$e^{j(3\omega_0 t)}\, Y(x) \quad \ldots \text{ etc.}$$

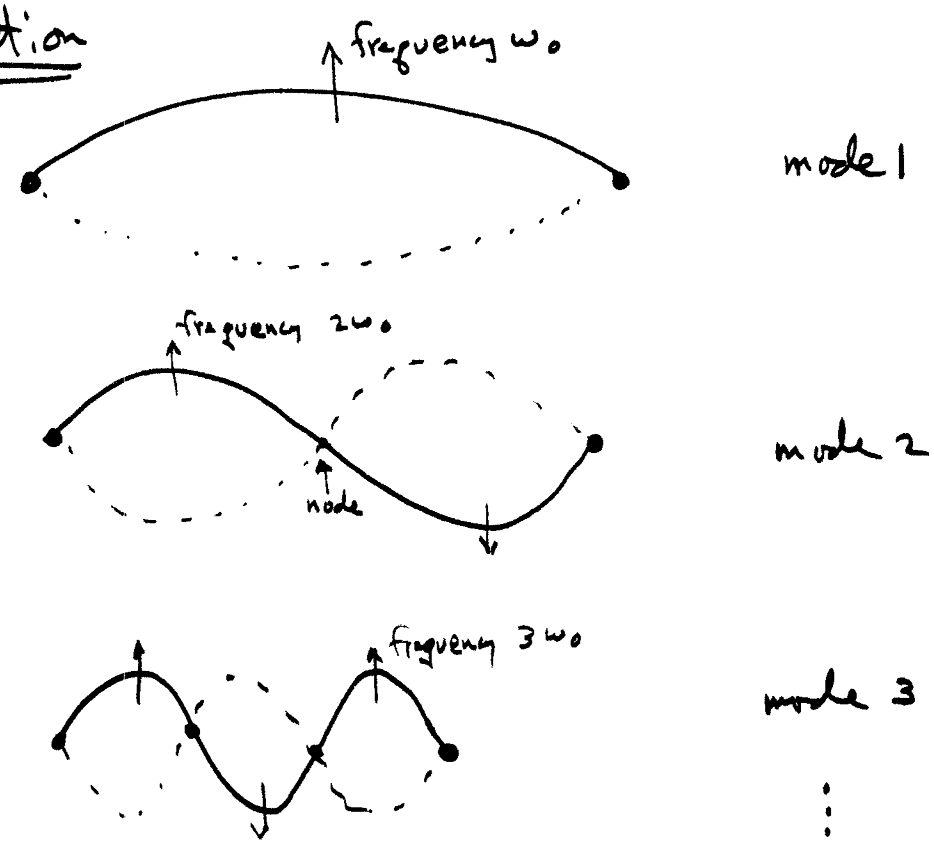
these lead to solns.

$$e^{jk\omega_0 t} \sin(k\pi x/L)$$

Sums of solutions are also solutions (<u>linear</u> eqn.)
so general soln. is

$$\boxed{\sum_{k=1}^{\infty} \underset{\substack{\uparrow \\ \text{arbitrary}}}{c_k}\, e^{jk\omega_0 t} \sin(k\pi x/L)}$$

<u>Interpretation</u>



$$\boxed{\sum_{k=1}^{\infty} c_k \, e^{jk\omega_0 t} \sin(k\pi x/L)}$$

can be any initial shape $\Rightarrow$ Fourier series can represent any function!

Back to general solution, Let $\underline{c=1}$ for now

$$y(x,t) = f(x+t) + g(x-t)$$

Usually, we are given $y(x,0)$ & $y'(x,0)$

differentiate

$$\begin{cases} f(x) + g(x) = F(x) & \leftarrow \text{given } \underline{\text{initial shape}} \\ f'(x) - g'(x) = G(x) & \leftarrow \text{given } \underline{\text{initial velocity}} \end{cases}$$

$$\underline{f'(x) + g'(x) = F'(x)}$$

add & subtract,

$$\begin{cases} f'(x) = \frac{1}{2}\left[F'(x) + G(x)\right] \\ g'(x) = \frac{1}{2}\left[F'(x) - G(x)\right] \end{cases}$$

$$\begin{cases} f(x) = \frac{1}{2}\left[F(x) + \int_0^x G(\eta)\,d\eta\right] + C \\ g(x) = \frac{1}{2}\left[F(x) - \int_0^x G(\eta)\,d\eta\right] + D \end{cases}$$

& use

$$y(x,t) = f(x+t) + g(x-t)$$

$$y(x,t) = \frac{1}{2}\left[F(x+t) + F(x-t) + \int_{x-t}^{x+t} G(\eta)\,d\eta\right] + \cancel{E}$$

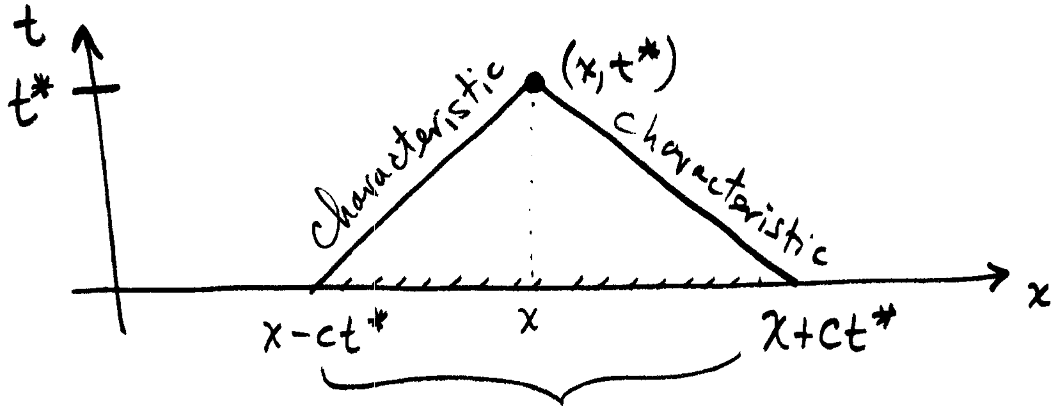
$\cancel{E} \to 0$ by $y(x,0) = F(x)$

for general c, this

d'Alembert's formula

$$\boxed{y(x,t) = \frac{1}{2}F(x+ct) + \frac{1}{2}F(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\eta)\,d\eta}$$

at time t^*, what values of F & G affect soln?



only soln. here can affect solution at (x, t^*)

this form can be used for calculation, but gets too complicated on finite string with reflections.