Chaos

Sources [GT96]
[Dickinson 91]
[EK 88]

Issues: Can "randomness" appear in deterministic systems without external noise?

Can "simple" systems behave in very complex ways?

Is certain kinds of behavior "Universal"?

We begin with a deceptively simple example.

LOGISTIC GROWTH

N Bacteria, suppose

$$\frac{dN}{dt} = kN \Rightarrow N = e^{kt}$$

EXPONENTIAL GROWTH
Malthus, 1798

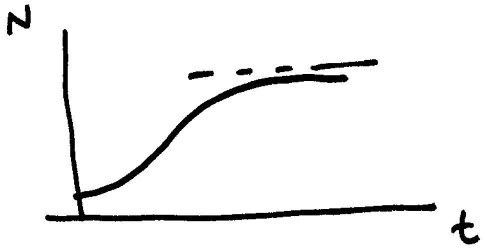
Limit on growth? suppose nutrient c

$$\frac{dN}{dt} = kcN$$

$$c = c_0 - \alpha N$$ (decreases with N)

$$\Rightarrow \frac{dN}{dt} = k(c_0 - \alpha N)N$$

LOGISTIC GROWTH
Verhulst, 1838



Normalized logistic difference equation:

$$x_{n+1} = a\, x_n(1-x_n)$$

by convention

$$\boxed{x_{n+1} = 4r\, x_n(1-x_n)} \leftarrow \text{"logistic map"}$$

restrict $0 \le x \le 1$, $0 < r \le 1$

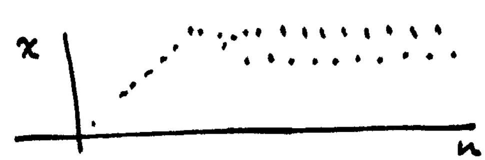
maps $[0,1] \to [0,1]$

Explore behavior:

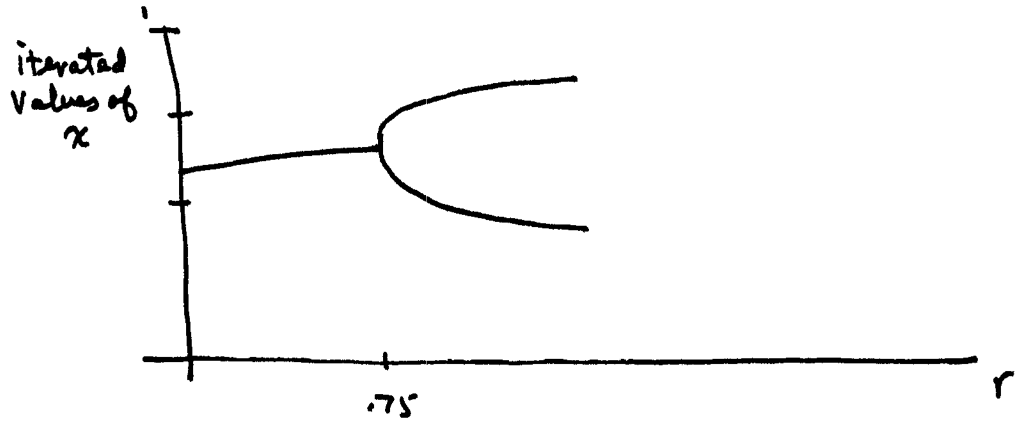
- small r, say $r = 0.24$ $\quad$ $x=0$ is a stable fixed pt.

- becomes unstable for $r > 0.25$
- for $r < 3/4$, $1 - 1/4r$ is a stable attractor.

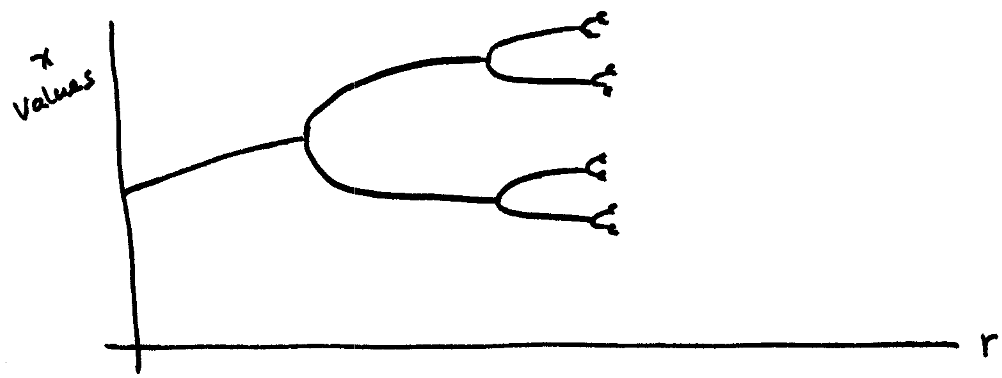
- for r slightly above $3/4$, we get a stable attractor of period 2



Plot values of x — after initial transients die down — vs. r ("control parameter")



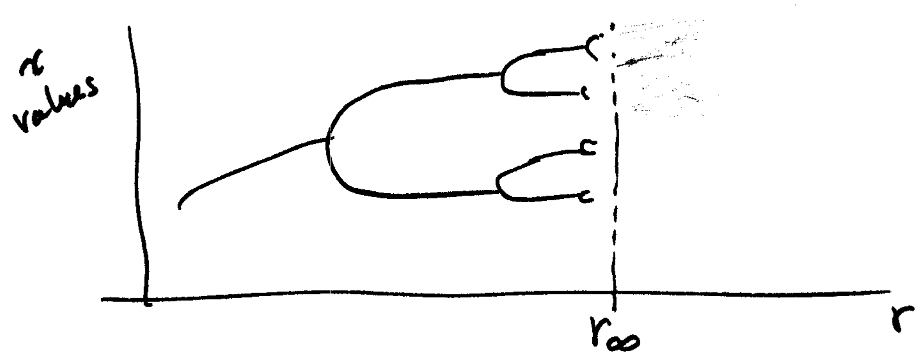
what happens if we increase r further?



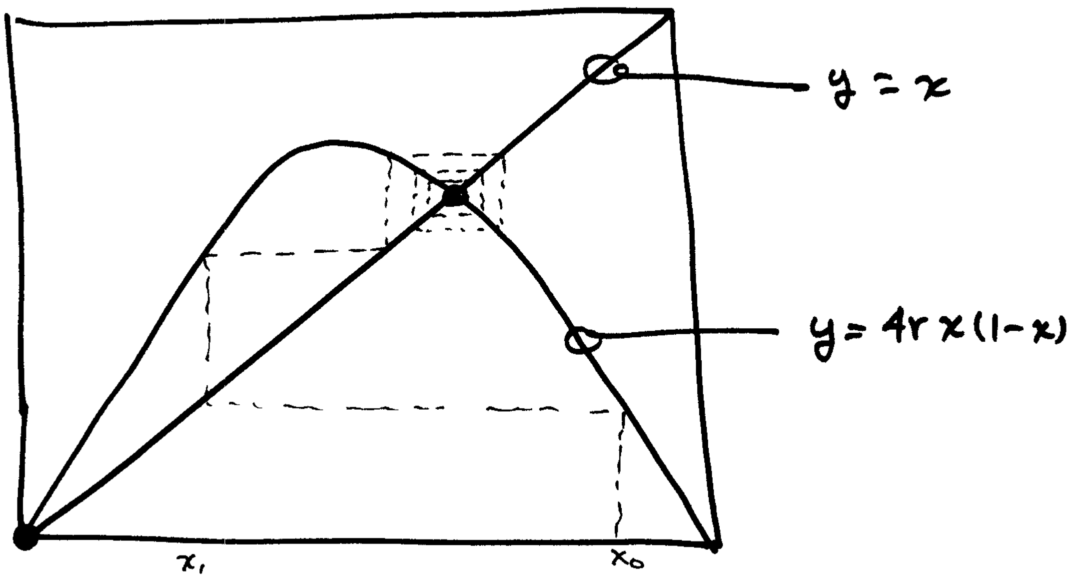
"Period doubling"

this ends! - and behavior becomes "chaotic" - at

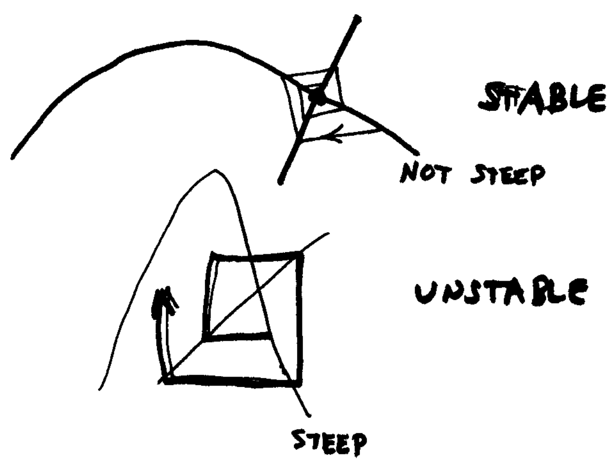
$$r = r_\infty = 0.89248641796 7\ldots$$



Graphical representation



STABILITY:
OF FIXED POINT



TURNS OUT THE CRITERION FOR STABILITY IS

$$|\text{Slope at fixed point}| < 1$$

i.e. origin is <u>unstable</u> unless

$$\text{Slope at } x=0 = 4r < 1 \Rightarrow \boxed{r < 1/4}$$

What about fixed points at intersection?

$$\cancel{x} = 4r\cancel{x}(1-x)$$

$$x = 1 - \frac{1}{4r}$$

Slope criterion is

$$|4r(1-2x)| < 1$$

Substitute

$$\left|4r\left(1-2\left[1-\tfrac{1}{4r}\right]\right)\right| < 1$$

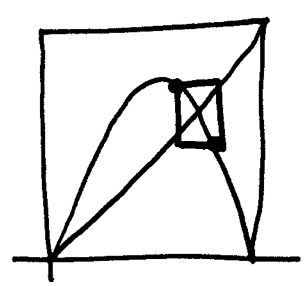
$$|2-4r| < 1$$

$$\pm(2-4r) < 1$$

$$\left.\begin{aligned} 2-4r<1 &\implies r>1/4 \\ -2+4r<1 &\implies r<3/4 \end{aligned}\right\}\ \text{as before}$$

What if $r > 3/4$?

look at $\underbrace{f(f(x))}_{f^{(2)}(x)}$ for period 2

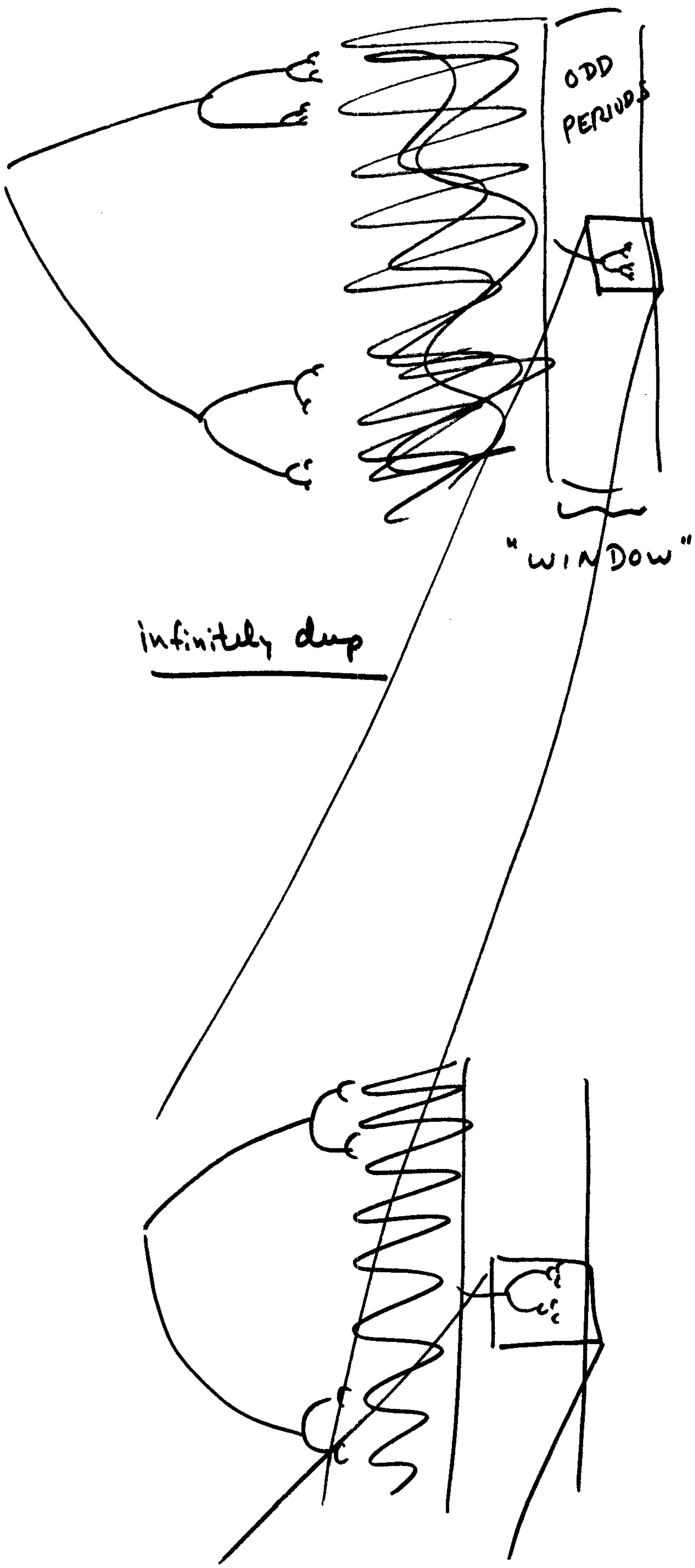


Period 2

$f^{(2)}(f^{(2)}(x))$ for period 4

... etc.

Surprises within chaotic regime:



Self-similar

(fractal)

3.3.6A

[GT96]

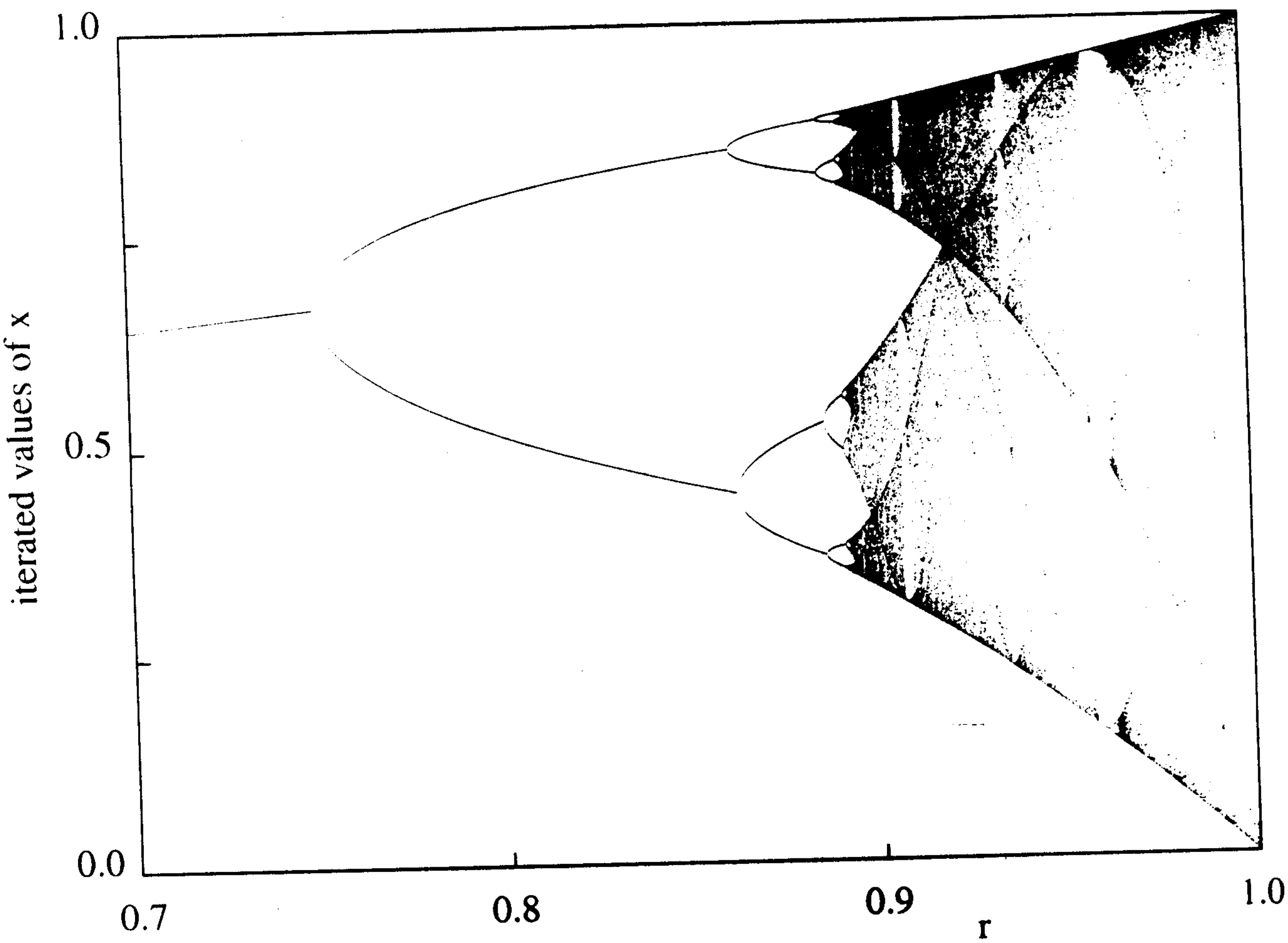
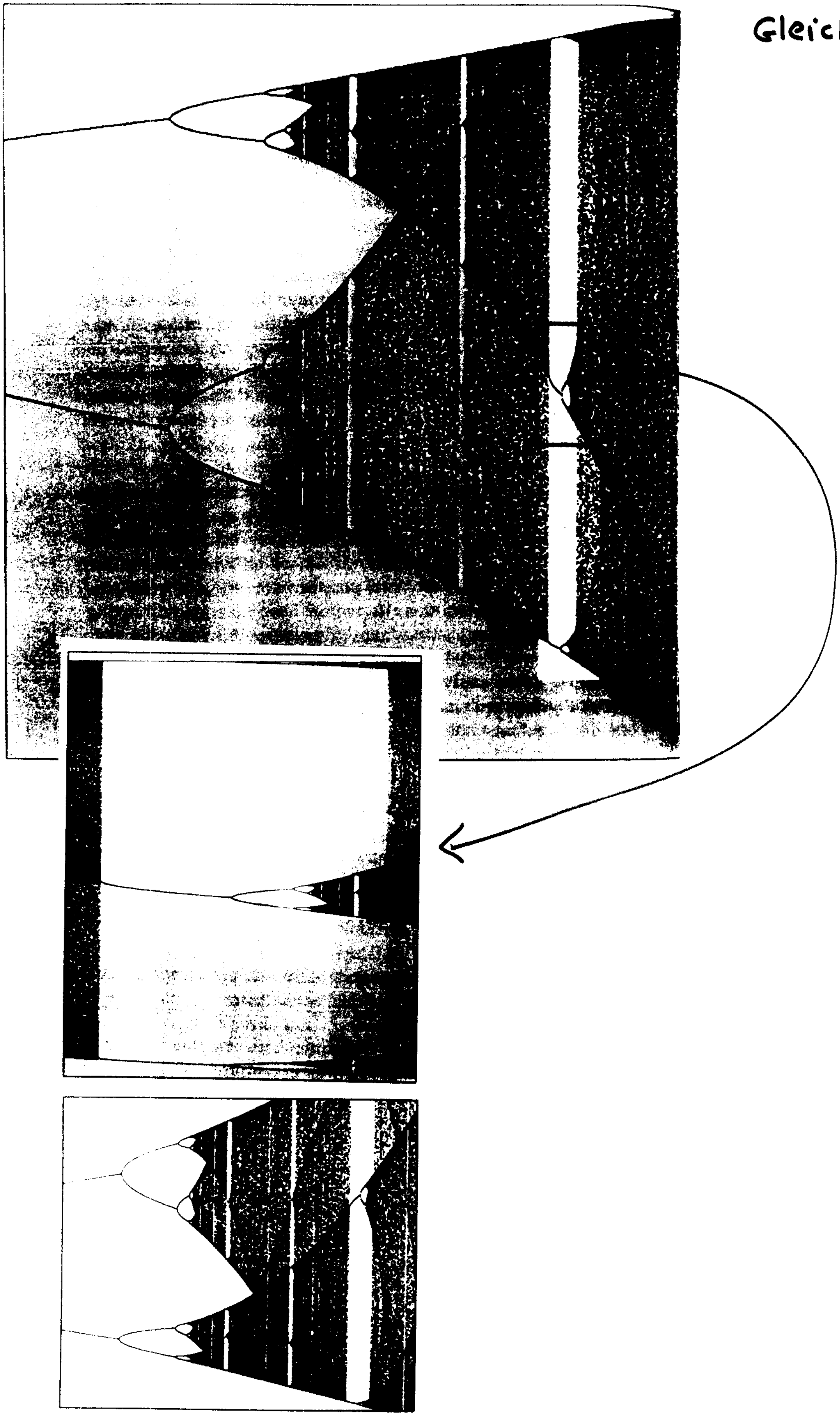


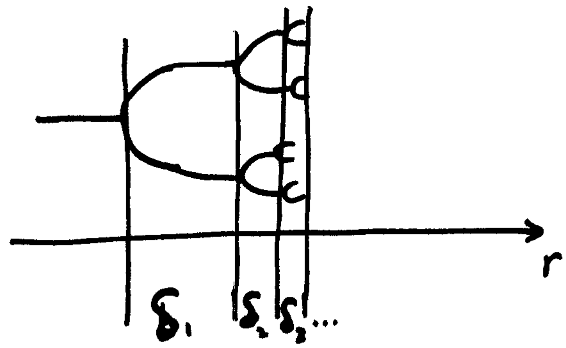
Fig. 6.2 Bifurcation diagram of the logistic map. For each value of r, the iterated values of x_n are plotted after the first 1000 iterations are discarded. Note the transition from periodic to chaotic behavior and the narrow windows of periodic behavior within the region of chaos.

3.3.6B
Gleick 87

Self-Similarity,
fractal



Pitchfork bifurcations & Approach to Chaos



get smaller & smaller

define $\delta = \lim \frac{\delta_k}{\delta_{k+1}}$

$\boxed{\delta = 4.669\,201\,609\,102\,991\ldots}$

Feigenbaum constant

Feigenbaum showed that δ is <u>independant</u> of the shape of the map, as long as map has $f'(x)=0$ & $f''(x)<0$ at max.

this "route to chaos" is <u>universal!</u>

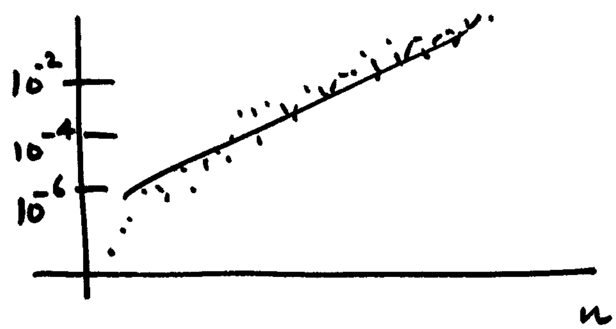
- Electrical circuits (ODES)
- water flow (PDEs)

→ Chaotic systems are sensitive to initial conditions

"Butterfly effect" in weather forecasting

<u>Example</u> Start logistic map at $x_0 = 0.5$
$x_0 = 0.5001$

Plot Δx_n = difference



observe <u>exponential</u> divergence of trajectory

$$|\Delta x_n| \approx |\Delta x_0| e^{\lambda n}$$

$$\lambda = \frac{1}{n} \ln \left| \frac{\Delta x_n}{\Delta x_0} \right|$$

to measure:

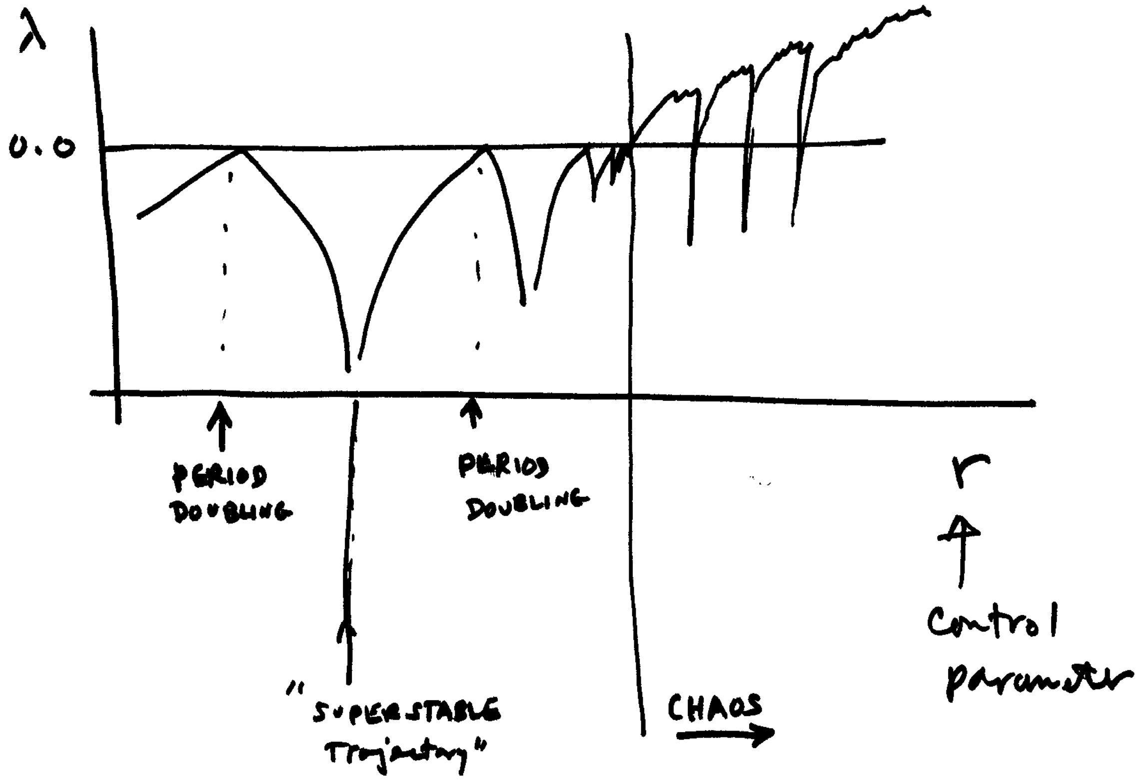
note $$\frac{\Delta x_n}{\Delta x_0} = \frac{\Delta x_n}{\Delta x_{n-1}} \frac{\Delta x_{n-1}}{\Delta x_{n-2}} \cdots \frac{\Delta x_1}{\Delta x_0}$$

$$\lambda = \frac{1}{n} \ln \left| \quad \right| = \frac{1}{n} \sum_{i=0}^{n-1} \ln \underbrace{\left| \frac{\Delta x_{i+1}}{\Delta x_i} \right|}_{\text{use } f'(x_i)}$$

because Δx_i is small

$$\boxed{\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|}$$

λ is called the <u>Lyapunov</u> exponent



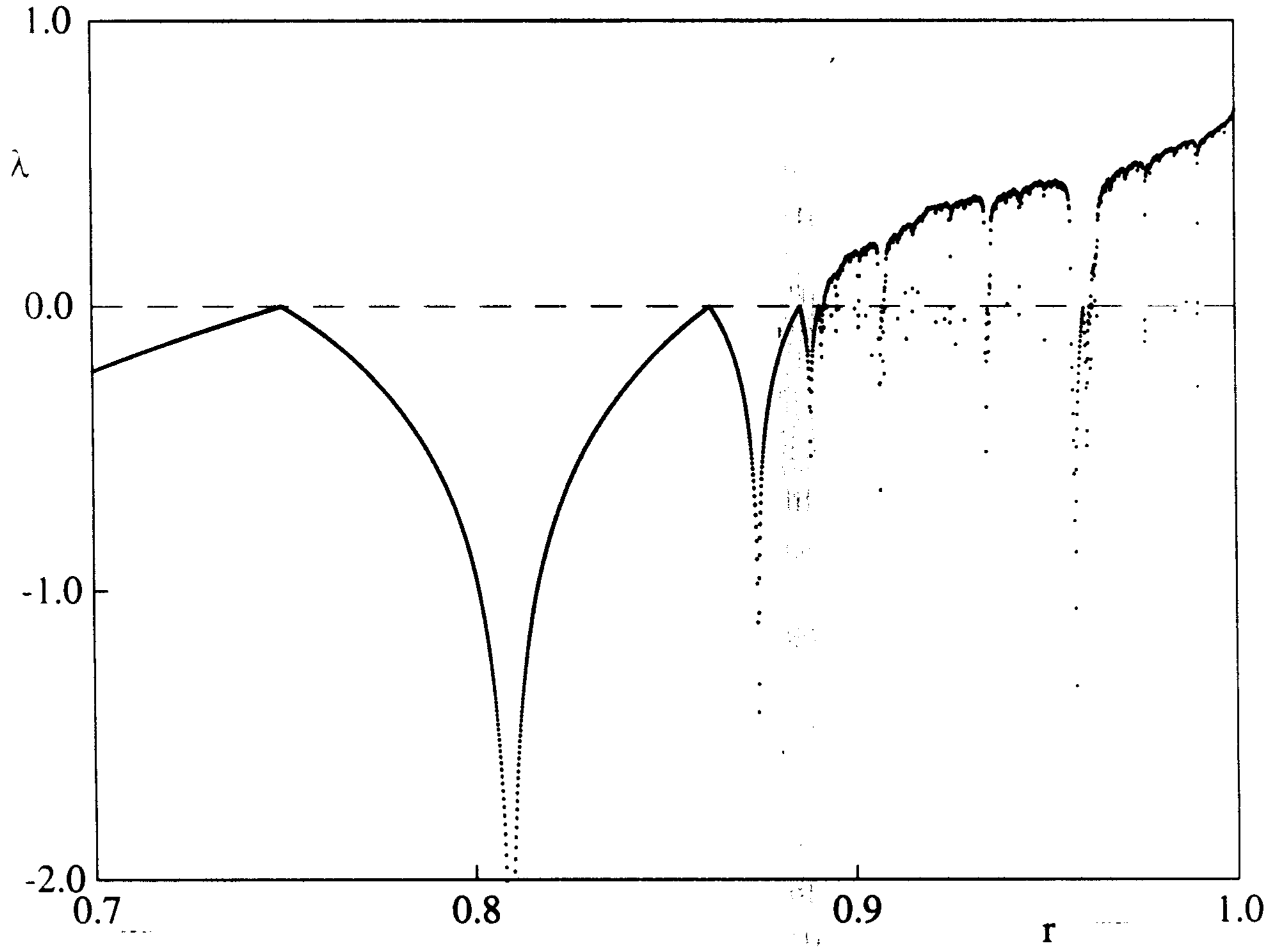


Fig. 6.9 The Lyapunov exponent calculated using the method in (6.19) as a function of the control parameter r. Compare the behavior of λ to the bifurcation diagram in Fig. 6.2. Note that $\lambda < 0$ for $r < 3/4$ and approaches zero at a period doubling bifurcation. A negative spike corresponds to a superstable trajectory. The onset of chaos is visible near $r = 0.892$, where λ first becomes positive. For $r > 0.892$, λ generally increases except for dips below zero whenever a periodic window occurs. Note the large dip due to the period 3 window near $r = 0.96$. For each value of r, the first 1000 iterations were discarded, and 10^5 values of $\ln|f'(x_n)|$ were used to determine λ.

3.3.10

100 Agents

CHAOTIC BEHAVIOR IN AN AUCTION-BASED MICRO-ECONOMIC MODEL

1992 See my home page for later version

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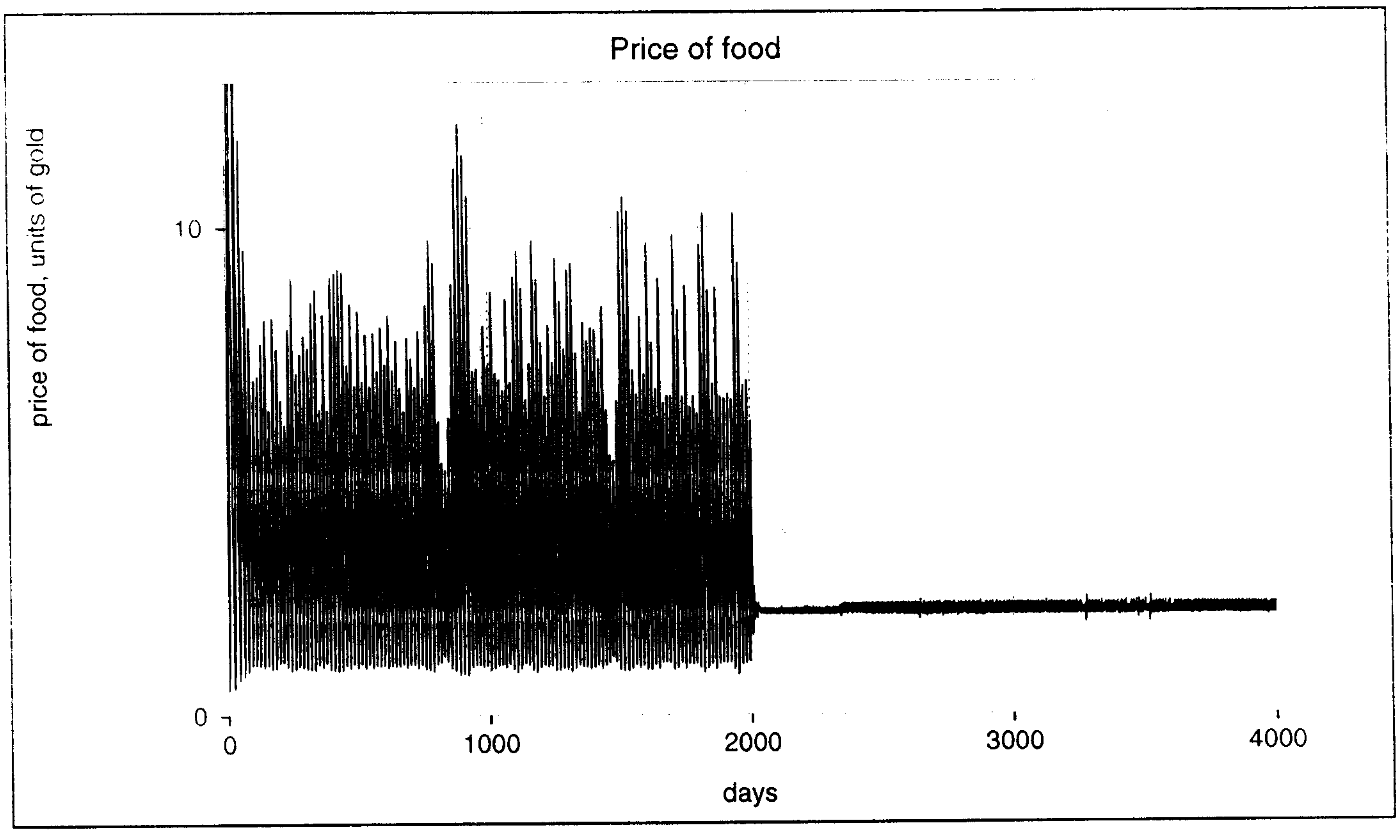
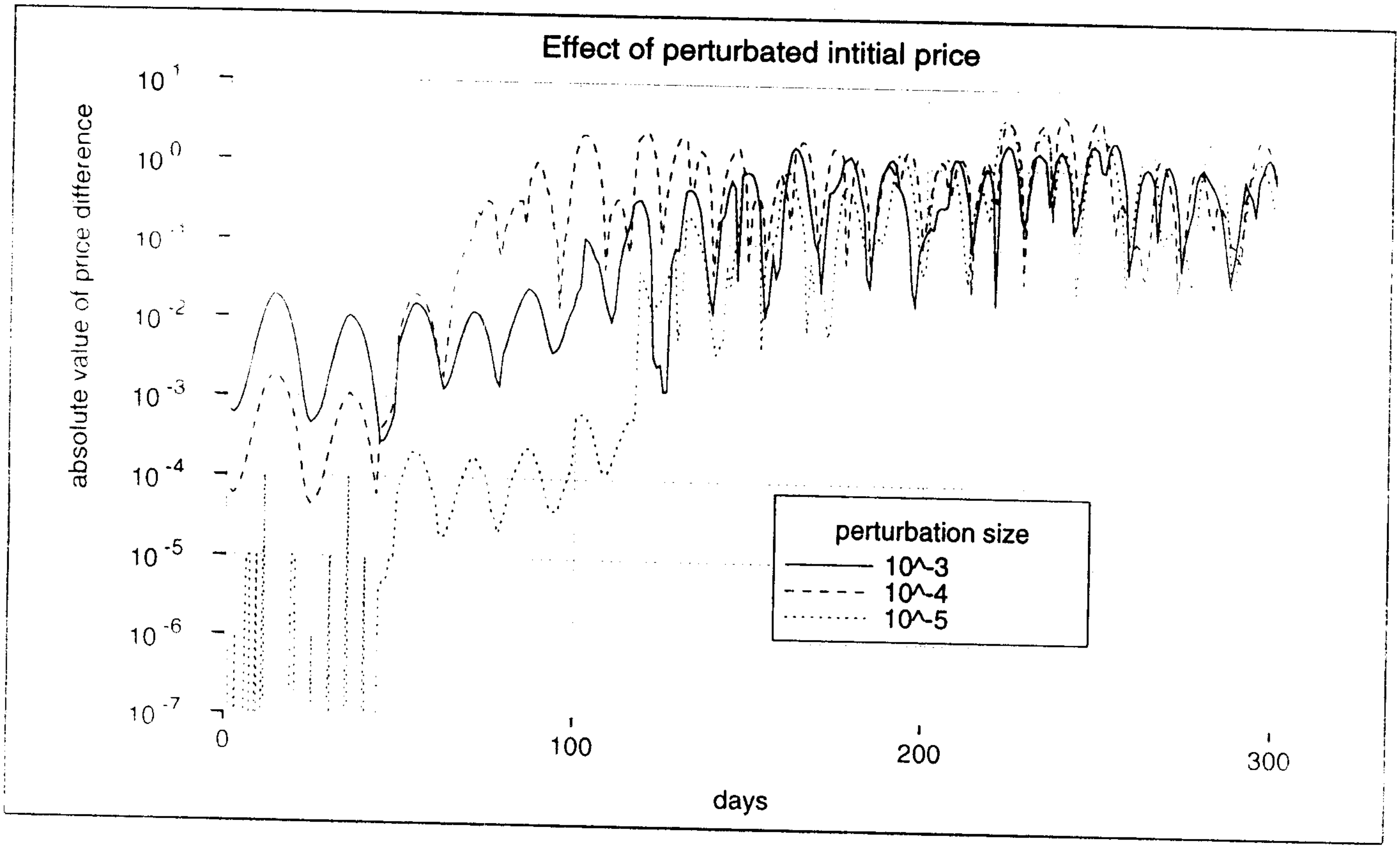
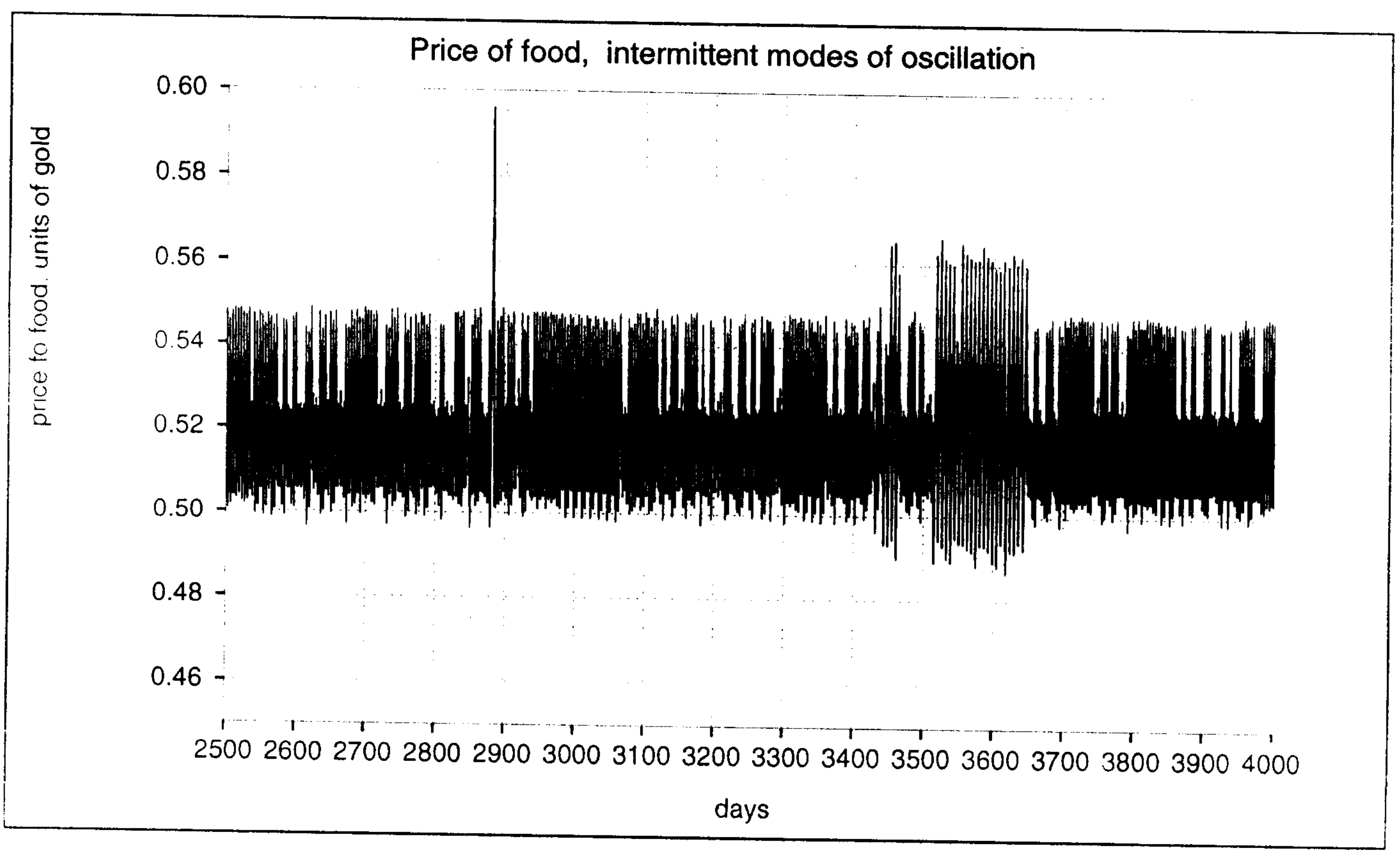




Effect of perturbated intitial price
absolute value of price difference
10^1
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
0
100
200
300
days
perturbation size
10^-3
10^-4
10^-5



Price of food, intermittent modes of oscillation
price fo food. units of gold
0.60
0.58
0.56
0.54
0.52
0.50
0.48
0.46
2500 2600 2700 2800 2900 3000 3100 3200 3300 3400 3500 3600 3700 3800 3900 4000
days

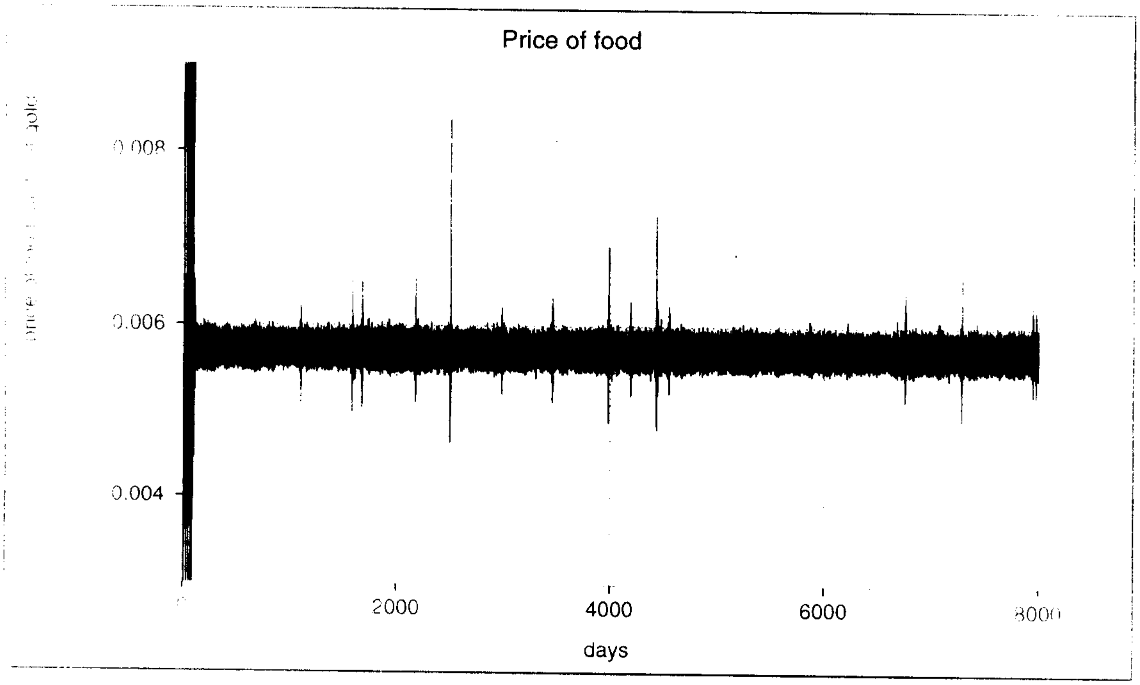


Fig. 14 An example of sporadic impulses in price. The case is that of 20 miners and 80 farmers, with prediction in the production decision. The utility function parameters are those of Fig. 5.

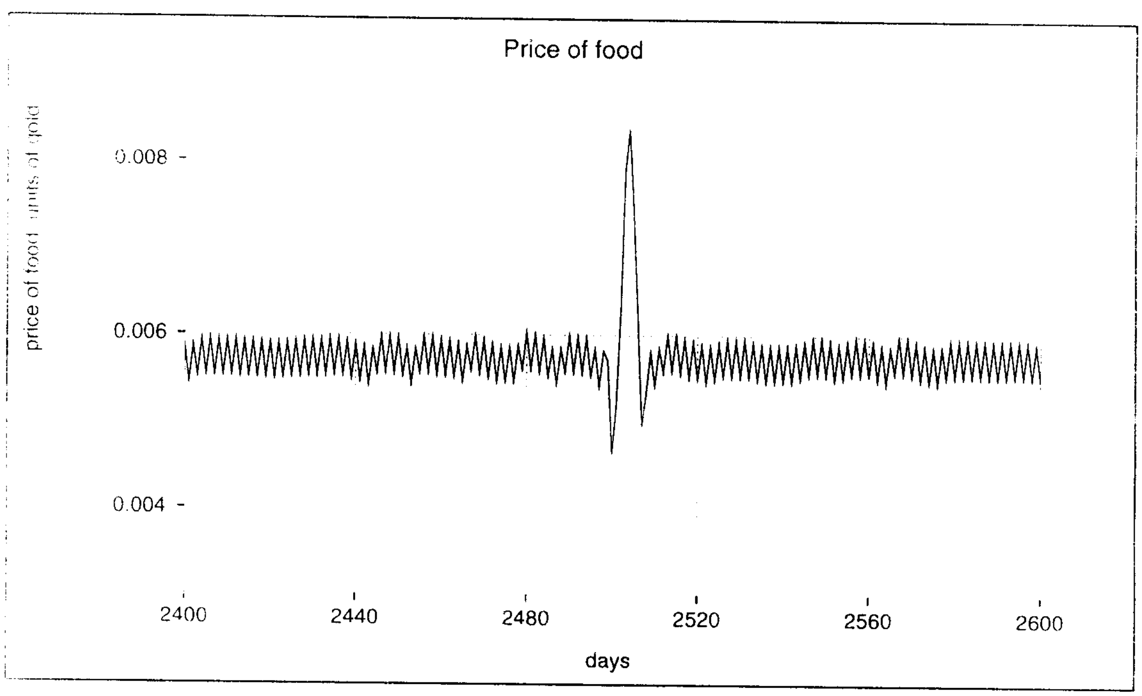


Fig. 15 Detail of Fig. 14.