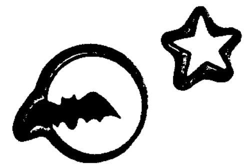
Ordinary Differential Equations

- One independent variable, often time t
- Formulated from physics, chemistry, biology, economics...
- Ubiquitous

→ Electrical circuits  $v(t) = L\frac{di(t)}{dt}$  $i(t) = C\frac{dV(t)}{dt}$

→ Orbital Mechanics $F = ma = \frac{d^2x(t)}{dt^2}$ 

→ chemical reactions — rate proportional to concentration

→ population dynamics — rate of change of population = function of population

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Population Dynamics (Example) [EK88]

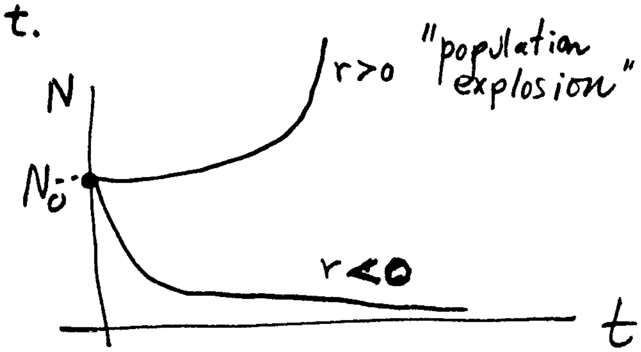
Malthus, 1998 — Let N = population = $N(t)$

$$\boxed{\frac{dN}{dt} = rN}$$

initial condition $\left.\frac{dN}{dt}\right|_{t=0} = rN_0 = r * (\text{initial population})$

$r = \text{const.}$

Solution $N(t) = N_0 e^{rt}$



How can we make this more realistic?

Crowding decreases rate of population growth

Simplest assumption is a linear factor $\left(1-\frac{N}{K}\right)$, where K is a limiting sustainable population.

this yields

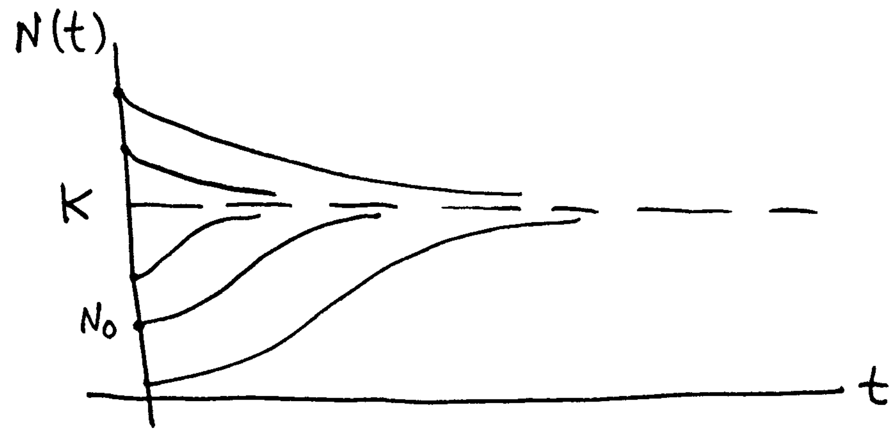
$$\boxed{\frac{dN}{dt} = r\left(1-\frac{N}{K}\right)N}$$

called

Logistic Growth (Verhulst, 1838)

Solution (analytical possible again)

$$N(t) = \underbrace{\left[\frac{1}{\frac{N_0}{K}+\left(1-\frac{N_0}{K}\right)e^{-rt}}\right]}_{\text{STARTS AT 1, heads towards } N_0/K,\text{ so } N\to K}\cdot N_0$$



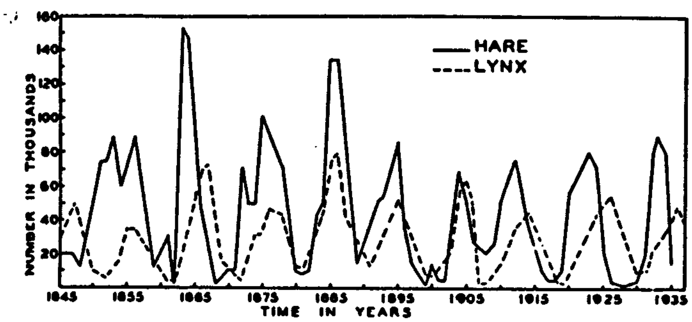
Comes up a lot in biological models of growth – yeast, bacteria, humans, ...

other models used to model growth of solid tumors, e.g.

these models have one dependent variable.
We often have more, e.g.

Predator-Prey Systems have been observed to oscillate for more than a century

Hudson Bay Company - traded in animal furs



[EK88]

Figure 6.3 Records dating back to the 1840s kept by the Hudson Bay Company. Their trade in pelts of the snowshoe hare and its predator the lynx reveals that the relative abundance of the two species undergoes dramatic cycles. The period of these cycles is roughly 10 years. [From E. P. Odum (1953), fig. 39.]

Mathematician V. Volterra studied observations of commercial fishing in Adriatic Sea:

During WW I, commercial fishing fell
Expected: fish increase
Actual: $\left\{ \begin{array}{l} \text{fish decrease} \\ \text{sharks increase} \end{array} \right.$
populations fluctuate

Volterra proposed model now known as the Volterra-Lotka Model to explain.

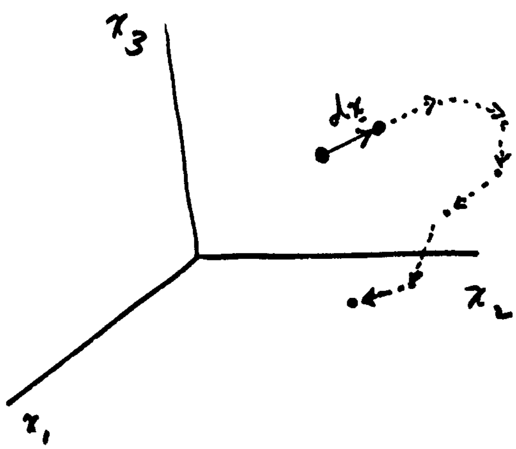
State-space Picture

Very productive way to look at O.D.E.'s

dependent variables $x_1, x_2, \cdots, x_n \rightarrow$ state vector $\underline{x}$ in n-space

transform equations to form

$$\underline{\dot{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} g_1(\underline{x},t) \\ g_2(\underline{x},t) \\ \vdots \\ g_n(\underline{x},t) \end{bmatrix} = \underline{g}(\underline{x},t)$$



infinitesimal Step $d\underline{x} = \underline{g}(\underline{x},t) \cdot dt$

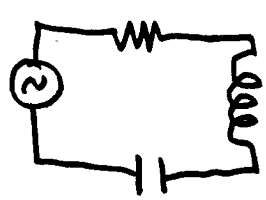
$\underline{g}$ tells how to move around state space

How can we put equations in state-space form?

Not unique.

Typical Example:

2^{nd} order linear ODE $\ddot{y} + \alpha\dot{y} + \beta y = f(t)$



, for example

NOT IN STATE-SPACE FORM

to put in state-space form:

Let $\begin{cases} x_1 = y \\ x_2 = \dot{y} \end{cases}$

then

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \ddot{y} = f(t) - \alpha \dot{y} - \beta y = \underbrace{f(t) - \alpha x_2 - \beta x_1}_{\text{fctn. of } x_1, x_2, t} \end{cases}$$

this is actually linear in x_1, x_2; a special case.
We write it

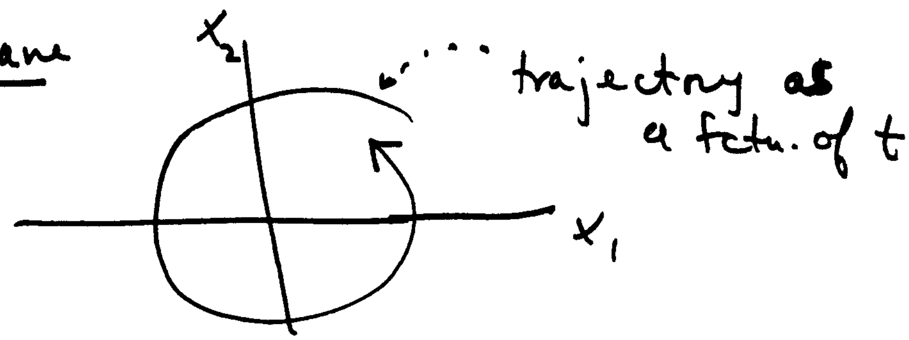
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\beta & -\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

or

$$\dot{\underline{x}} = \underbrace{\underline{\underline{A}}}_{\text{const. matrix}} \underline{x} + \underbrace{\underline{b}}_{\text{const. vector}} f(t) \qquad \boxed{\dot{\underline{x}} = g(\underline{x}, t)}$$

When state-space is actually only 2-dimensional, this is a very beautiful way to see what's happening.

Called phase-plane



Even in higher-dimensional systems, we can look at two coordinates at a time.

Volterra-Lotka Model

[Dickinson, *Systems*, 1991]
[EK88]

$x_1(t)$ = biomass of population of *predatory* species
$x_2(t)$ = " " " " *prey* "

relative rate of change of x_1 $= \dfrac{\dot{x}_1}{x_1} = \underbrace{\overset{>0}{b_{12}}\, x_2}_{\text{prey increases predators}} - \underbrace{\overset{>0}{a_1}}_{\text{no prey} \Rightarrow \text{predators die out}}$

- - - - - x_2 $= \dfrac{\dot{x}_2}{x_2} = \underbrace{\overset{>0}{a_2}}_{\text{no predators} \Rightarrow \text{prey grow}} - \underbrace{\overset{>0}{b_{21}}\, x_1}_{\text{predators decrease prey}}$

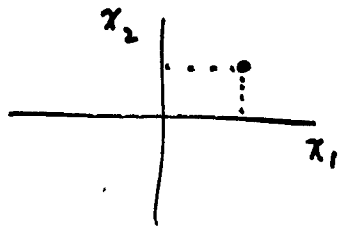
$$\left\{ \begin{aligned} \dot{x}_1 &= x_1 (b_{12} x_2 - a_1) \\ \dot{x}_2 &= x_2 (a_2 - b_{21} x_1) \end{aligned} \right\}$$ state-space form

Easy to find possible equilibria in this case:

set $\dot{\underline{x}} = 0$: $\quad \dot{x}_1 = 0 = x_1 (b_{12} x_2 - a_1)$
$\qquad \dot{x}_2 = 0 = x_2 (a_2 - b_{21} x_1)$

$x_1 = 0$ or $x_2 = 0$ not acceptable in model.

$$\therefore \left\{ \begin{aligned} x_2 &= a_1 / b_{12} \\ x_1 &= a_2 / b_{21} \end{aligned} \right.$$



unique equilibrium pt.

Dynamic Behavior

We can sometimes deduce a lot from phase-plane analysis (working with this example in 2-dimensions).

from state-space formula

small change in $\underline{x}$ over time dt

$$\begin{cases} dx_1 = x_1 (b_{12} x_2 - a_1)\, dt \\ dx_2 = x_2 (a_2 - b_{21} x_1)\, dt \end{cases}$$

implies
$$\frac{dx_2}{dx_1} = \frac{x_2}{x_1} \left(\frac{a_2 - b_{21} x_1}{b_{12} x_2 - a_1} \right)$$

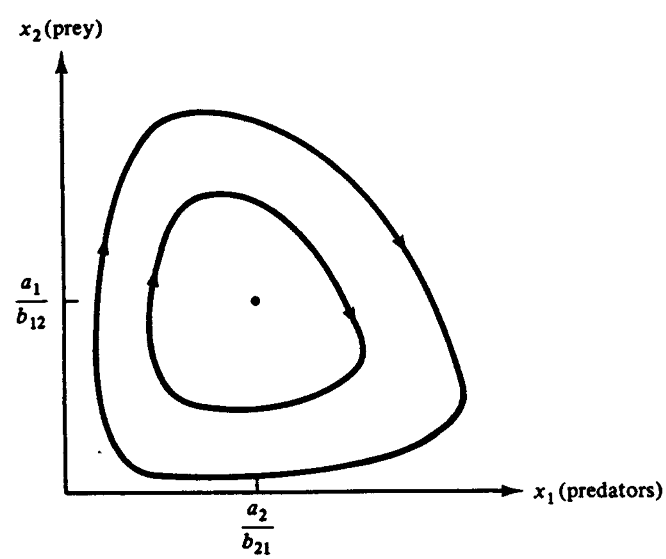
this differential eqn. can be integrated analytically, yielding

$$b_{12} x_2 - a_1 \ln x_2 + C = -b_{21} x_1 + a_2 \ln x_1$$

where C = arbitrary constant.

(just differentiate to check this claim)

this defines a <u>family</u> of <u>closed</u> <u>curves</u> in $x_1 - x_2$ plane (how to compute?)

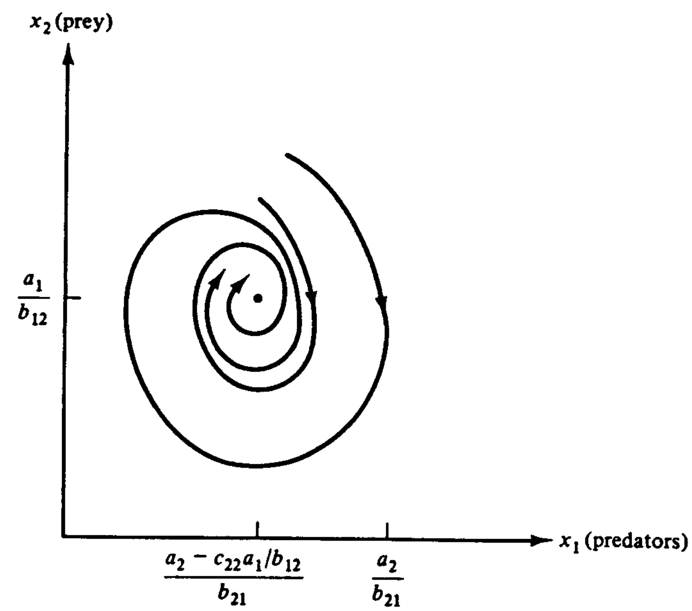


one periodic solution for each const. of integration C

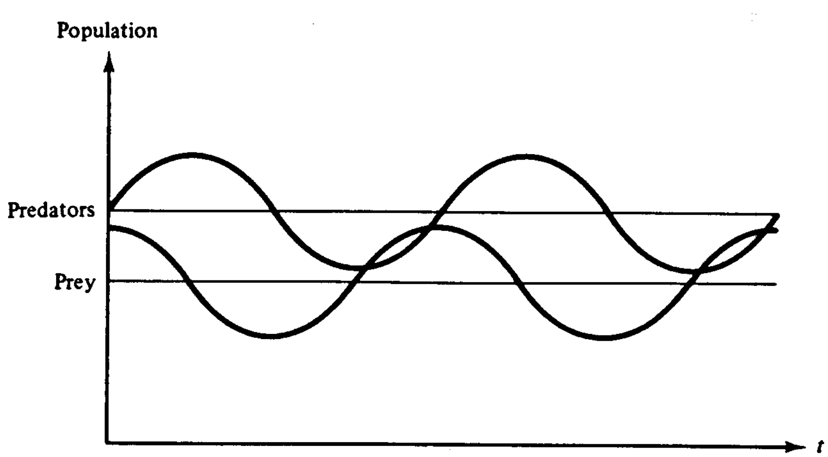
A reasonable variation on the model (always the game) leads to reasonable change in behavior:

Model fact that prey is self-limiting; as prey population increases, ultimately rate of growth decreases because food supply is limited. Add term in $\dot{x}_2$

$$\frac{\dot{x}_2}{x_2} = a_2 - b_{21} x_1 \underbrace{- c_{22} x_2}_{\text{self-limiting term}}$$



all trajectories now converge to "Stable focus"

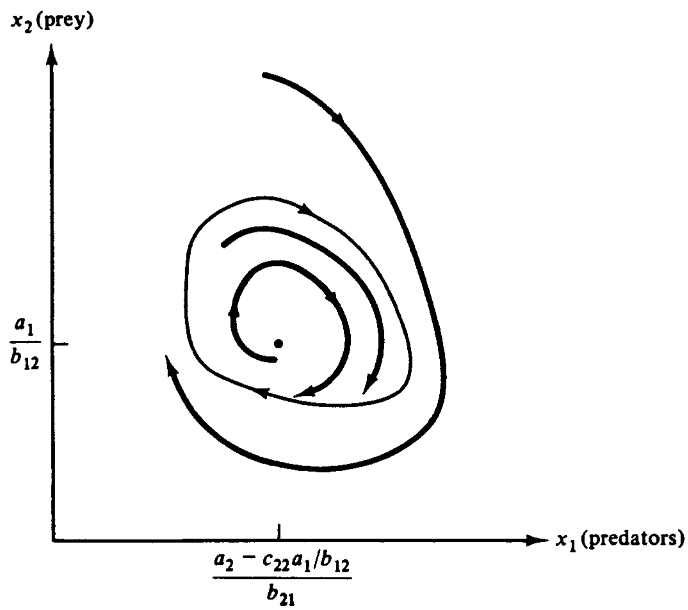


Population fluctuations are roughly $\pi/2$ out of phase

A corresponding destabilizing modification is to replace $x_2(t)$ by a delayed version, say $x_2(t-T)$. This gets us outside the realm of differential equations. But we can approximate with

$$x_2(t-T) \approx x_2 - T\dot{x}_2 \qquad \text{(Taylor series again)}$$

Leads to a very interesting phenomenon



closed curve is the limit cycle

Does this exhaust the kinds of behavior in dynamic systems?

Chaos strange attractors, etc.

No stable equilibria, No limit cycles,
but space-filling trajectories

Famous Example Lorenz Attractor

arose first in fluid dynamics for weather modeling

3-dimensional state space
$$\begin{cases} \dot{x}_1 = \alpha(x_2 - x_1) \\ \dot{x}_2 = (1+\beta - x_3)\,x_1 - x_2 \\ \dot{x}_3 = x_1 x_2 - \gamma x_3 \end{cases}$$