COS429: COMPUTER VISION
CAMERAS AND PROJECTIONS (2 lectures)

• Pinhole cameras
• Analytical Euclidean geometry
• The intrinsic parameters of a camera
• The extrinsic parameters of a camera
• Camera calibration
• Least-squares techniques

Reading: Chapters 1 – 3, Forsyth & Ponce

Many of the slides in this lecture are courtesy to Prof. J. Ponce
Some history...

Milestones:
• Leonardo da Vinci (1452-1519): first record of camera obscura
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• Daguerréotypes (1839)
• Photographic Film (Eastman, 1889)
• Cinema (Lumière Brothers, 1895)
• Color Photography (Lumière Brothers, 1908)
• Television (Baird, Farnsworth, Zworykin, 1920s)

Photography (Niepce, “La Table Servie,” 1822)
Let’s also not forget…

Motzu  
(468-376 BC)

Aristotle  
(384-322 BC)

Ibn al-Haitham  
(965-1040)
Pinhole perspective projection
Distant objects are smaller
Parallel lines meet

Common to draw image plane *in front* of the focal point. Moving the image plane merely scales the image.
Vanishing points

• Each set of parallel lines meets at a different point
  – The *vanishing point* for this direction

• Sets of parallel lines on the same plane lead to *collinear* vanishing points.
  – The line is called the *horizon* for that plane
Properties of Projection

• Points project to points
• Lines project to lines
• Planes project to the whole image or a half image
• Angles are not preserved
• Degenerate cases
  – Line through focal point projects to a point.
  – Plane through focal point projects to line
  – Plane perpendicular to image plane projects to part of the image (with horizon).
Pinhole Perspective Equation

\[ \begin{align*}
\frac{x}{f} &= \frac{y}{f'} = \frac{z}{f''} \\
\end{align*} \]

NOTE: \( z \) is always negative..
Affine projection models: Weak perspective projection

When the scene depth is small compared its distance from the Camera, \( m \) can be taken constant: weak perspective projection.
Affine projection models: Orthographic projection

When the camera is at a (roughly constant) distance from the scene, take $m=1$. 
Pros and Cons of These Models

• Weak perspective much simpler math.
  – Accurate when object is small and distant.
  – Most useful for recognition.

• Pinhole perspective much more accurate for scenes.
  – Used in structure from motion.

• When accuracy really matters, must model real cameras.
Diffraction effects in pinhole cameras.

Shrinking pinhole size

Use a lens!
Quantitative Measurements and Calibration

Euclidean Geometry
Euclidean Coordinate Systems
Coordinate Changes: Pure Translations

\[ \overrightarrow{O_B P} = \overrightarrow{O_B O_A} + \overrightarrow{O_A P} , \quad \overrightarrow{B P} = \overrightarrow{A P} + \overrightarrow{B O_A} \]
Coordinate Changes: Pure Rotations
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Coordinate Changes: Pure Rotations
Coordinate Changes: Rotations about the $z$ Axis

\[
\begin{bmatrix}
100 \\
0 \\
\cos \theta \\
\sin \theta
\end{bmatrix}
\]

\[
R_B^A =
\begin{bmatrix}
100 \\
0 \\
\cos \theta \\
\sin \theta
\end{bmatrix}
\]
A rotation matrix is characterized by the following properties:

- Its inverse is equal to its transpose, and
- its determinant is equal to 1.

Or equivalently:

- Its rows (or columns) form a right-handed orthonormal coordinate system.
Coordinate Changes: Rigid Transformations
Block Matrix Multiplication

What is $AB$?

Homogeneous Representation of Rigid Transformations
Rigid Transformations as Mappings

\[
^{F}P' = R^{F}P + t \iff \begin{pmatrix} ^{F}P' \\ 1 \end{pmatrix} = \begin{pmatrix} R \\ 0^{T} \end{pmatrix} \begin{pmatrix} t \\ 1 \end{pmatrix} \begin{pmatrix} ^{F}P \\ 1 \end{pmatrix}
\]
Rigid Transformations as Mappings: Rotation about the $k$ Axis

\[ ^F P' = \mathcal{R}^F P, \quad \text{where} \quad \mathcal{R} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
Pinhole Perspective Equation

\[
\begin{align*}
x' &= f' \frac{x}{z} \\
y' &= f' \frac{y}{z}
\end{align*}
\]
The Intrinsic Parameters of a Camera

Units:

\( k, l \) : pixel/m

\( f \) : m

\( \alpha, \beta \) : pixel

\[
\begin{align*}
\hat{u} &= \frac{x}{z} \\
\hat{v} &= \frac{y}{z}
\end{align*}
\]

\( \iff \)

\[
\hat{p} = \frac{1}{z} \begin{pmatrix} \text{Id} & 0 \end{pmatrix} \begin{pmatrix} P \end{pmatrix}
\]

Physical Image Coordinates

Normalized Image Coordinates

\[
\begin{align*}
u &= k f \frac{x}{z} \\
v &= l f \frac{y}{z}
\end{align*} \quad \begin{align*}
u &= \alpha \frac{x}{z} + u_0 \\
v &= \beta \frac{y}{z} + v_0
\end{align*} \quad \begin{align*}
u &= \alpha \frac{x}{z} - \alpha \cot \theta \frac{y}{z} + u_0 \\
v &= \frac{\beta}{\sin \theta} \frac{y}{z} + v_0
\end{align*}
\]
The Intrinsic Parameters of a Camera

Calibration Matrix

\[ p = \mathcal{K}\hat{p}, \quad \text{where} \quad p = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{K} \overset{\text{def}}{=} \begin{pmatrix} \alpha & -\alpha \cot \theta & u_0 \\ 0 & \frac{\beta}{\sin \theta} & v_0 \\ 0 & 0 & 1 \end{pmatrix} \]

The Perspective Projection Equation

\[ p = \frac{1}{z} \mathcal{M}P, \quad \text{where} \quad \mathcal{M} \overset{\text{def}}{=} \left( \begin{array}{cc} \mathcal{K} & 0 \end{array} \right) \]
The Extrinsic Parameters of a Camera

- When the camera frame \((C)\) is different from the world frame \((W)\),

\[
\begin{pmatrix}
  C \mathbf{P} \\
  1
\end{pmatrix} = \begin{pmatrix}
  C \mathbf{R} & C \mathbf{O}_W \\
  0^T & 1
\end{pmatrix}
\begin{pmatrix}
  W \mathbf{P} \\
  1
\end{pmatrix}.
\]

- Thus,

\[
\mathbf{p} = \frac{1}{z} \mathbf{M} \mathbf{P},
\]

where

\[
\begin{align*}
\mathbf{M} &= \mathbf{K} \begin{pmatrix} \mathbf{R} & \mathbf{t} \end{pmatrix}, \\
\mathbf{R} &= C_W \mathbf{R}, \\
\mathbf{t} &= C \mathbf{O}_W, \\
\mathbf{P} &= \begin{pmatrix} W \mathbf{P} \\
1\end{pmatrix}.
\end{align*}
\]

- Note: \(z\) is not independent of \(\mathbf{M}\) and \(\mathbf{P}\):

\[
\mathbf{M} = \begin{pmatrix}
  m_1^T \\
  m_2^T \\
  m_3^T
\end{pmatrix} \quad \Rightarrow \quad z = m_3 \cdot \mathbf{P}, \quad \text{or} \quad \begin{align*}
u &= \frac{m_1 \cdot \mathbf{P}}{m_3 \cdot \mathbf{P}}, \\
v &= \frac{m_2 \cdot \mathbf{P}}{m_3 \cdot \mathbf{P}}.
\end{align*}
\]
Explicit Form of the Projection Matrix

\[
M = \begin{pmatrix}
\alpha r_1^T - \alpha \cot \theta r_2^T + u_0 r_3^T & \alpha t_x - \alpha \cot \theta t_y + u_0 t_z \\
\frac{\beta}{\sin \theta} r_2^T + v_0 r_3^T & \frac{\beta}{\sin \theta} t_y + v_0 t_z \\
\frac{r_3^T}{\sin \theta} & t_z
\end{pmatrix}
\]

Note: If \( M = (A \ b) \) then \( |a_3| = 1 \).

Replacing \( M \) by \( \lambda M \) in

\[
\begin{cases}
u = \frac{m_1 \cdot P}{m_3 \cdot P} \\
v = \frac{m_2 \cdot P}{m_3 \cdot P}
\end{cases}
\]

does not change \( u \) and \( v \).

\( M \) is only defined up to scale in this setting!!
Theorem (Faugeras, 1993)

Let \( \mathcal{M} = (\mathcal{A} \ b) \) be a \( 3 \times 4 \) matrix and let \( \mathbf{a}_i^T \ (i = 1, 2, 3) \) denote the rows of the matrix \( \mathcal{A} \) formed by the three leftmost columns of \( \mathcal{M} \).

- A necessary and sufficient condition for \( \mathcal{M} \) to be a perspective projection matrix is that \( \text{Det}(\mathcal{A}) \neq 0 \).

- A necessary and sufficient condition for \( \mathcal{M} \) to be a zero-skew perspective projection matrix is that \( \text{Det}(\mathcal{A}) \neq 0 \) and
  \[
  (\mathbf{a}_1 \times \mathbf{a}_3) \cdot (\mathbf{a}_2 \times \mathbf{a}_3) = 0.
  \]

- A necessary and sufficient condition for \( \mathcal{M} \) to be a perspective projection matrix with zero skew and unit aspect-ratio is that \( \text{Det}(\mathcal{A}) \neq 0 \) and
  \[
  \begin{cases}
  (\mathbf{a}_1 \times \mathbf{a}_3) \cdot (\mathbf{a}_2 \times \mathbf{a}_3) = 0, \\
  (\mathbf{a}_1 \times \mathbf{a}_3) \cdot (\mathbf{a}_1 \times \mathbf{a}_3) = (\mathbf{a}_2 \times \mathbf{a}_3) \cdot (\mathbf{a}_2 \times \mathbf{a}_3).
  \end{cases}
  \]
Quantitative Measurements and the Calibration Problem
Calibration Procedure

- Calibration target: 2 planes at right angle with checkerboard (Tsai grid)
- We know positions of corners of grid with respect to a coordinate system of the target
- Obtain from images the corners
- Using the equations (relating pixel coordinates to world coordinates) we obtain the camera parameters (the internal parameters and the external (pose) as a side effect)
Estimation procedure

• First estimate $M$ from corresponding image points and scene points (solving homogeneous equation)
• Second decompose $M$ into internal and external parameters
• Use estimated parameters as starting point to solve calibration parameters non-linearly.
Homogeneous Linear Systems

Square system:
- unique solution: 0
- unless $\text{Det}(A)=0$

Rectangular system ??
- 0 is always a solution

Minimize $|Ax|^2$ under the constraint $|x|^2=1$
How do you solve overconstrained homogeneous linear equations ??

\[ E = |Ux|^2 = x^T (U^TU)x \]

- Orthonormal basis of eigenvectors: \( e_1, \ldots, e_q \).

- Associated eigenvalues: \( 0 \leq \lambda_1 \leq \ldots \leq \lambda_q \).

- Any vector can be written as
  \[
  x = \mu_1 e_1 + \ldots + \mu_q e_q
  \]
  for some \( \mu_i \) \( (i = 1, \ldots, q) \) such that \( \mu_1^2 + \ldots + \mu_q^2 = 1 \).

\[ E(x) - E(e_1) = x^T (U^TU)x - e_1^T (U^TU)e_1 \]
\[ = \lambda_1 \mu_1^2 + \ldots + \lambda_q \mu_q^2 - \lambda_1 \]
\[ \geq \lambda_1 (\mu_1^2 + \ldots + \mu_q^2 - 1) = 0 \]

The solution is \( e_1 \).
Example: Line Fitting

Problem: minimize

\[ E(a, b, d) = \sum_{i=1}^{n} (ax_i + by_i - d)^2 \]

with respect to \((a,b,d)\).

- Minimize \(E\) with respect to \(d\):

\[
\frac{\partial E}{\partial d} = 0 \implies d = \frac{1}{n} \sum_{i=1}^{n} ax_i + by_i = \bar{a}x + \bar{b}y
\]

- Minimize \(E\) with respect to \(a, b\):

\[
E = \sum_{i=1}^{n} [a(x_i - \bar{x}) + b(y_i - \bar{y})]^2 = |\mathcal{U}n|^2 \quad \text{where} \quad \mathcal{U} = \begin{pmatrix} x_1 - \bar{x} & y_1 - \bar{y} \\ \vdots & \vdots \\ x_n - \bar{x} & y_n - \bar{y} \end{pmatrix}
\]

- Done !!
Note:

\[ U^T U = \begin{pmatrix}
\sum_{i=1}^{n} x_i^2 - nx^2 & \sum_{i=1}^{n} x_i y_i - nxy \\
\sum_{i=1}^{n} x_i y_i - nxy & \sum_{i=1}^{n} y_i^2 - ny^2
\end{pmatrix} \]

- Matrix of second moments of inertia
- Axis of least inertia
Linear Camera Calibration

Given $n$ points $P_1, \ldots, P_n$ with known positions and their images $p_1, \ldots, p_n$

\[
\begin{pmatrix}
u_i \\
v_i
\end{pmatrix} = \begin{pmatrix}
m_1 \cdot P_i \\
m_3 \cdot P_i \\
m_2 \cdot P_i \\
m_3 \cdot P_i
\end{pmatrix} \iff \begin{pmatrix}
m_1 - u_i m_3 \\
m_2 - v_i m_3
\end{pmatrix} P_i = 0
\]

\[
\mathcal{P} m = 0 \text{ with } \mathcal{P} \overset{\text{def}}{=} \begin{pmatrix}
P_1^T & 0^T & -u_1 P_1^T \\
0^T & P_1^T & -v_1 P_1^T \\
\vdots & \vdots & \vdots \\
P_n^T & 0^T & -u_n P_n^T \\
0^T & P_n^T & -v_n P_n^T
\end{pmatrix} \text{ and } m \overset{\text{def}}{=} \begin{pmatrix}
m_1 \\
m_2 \\
m_3
\end{pmatrix} = 0
\]
Once $M$ is known, you still got to recover the intrinsic and extrinsic parameters !!!

This is a decomposition problem, not an estimation problem.

$$\begin{pmatrix} \rho \\ \mathcal{M} \end{pmatrix} = \begin{pmatrix} \alpha r_1^T - \alpha \cot \theta r_2^T + u_0 r_3^T & \alpha t_x - \alpha \cot \theta t_y + u_0 t_z \\ \frac{\beta}{\sin \theta} r_2^T + v_0 r_3^T & \frac{\beta}{\sin \theta} t_y + v_0 t_z \\ r_3^T & t_z \end{pmatrix}$$

- Intrinsic parameters
- Extrinsic parameters
Degenerate Point Configurations

Are there other solutions besides $M$?

$$0 = \mathcal{P}l = \begin{pmatrix} P_1^T & 0^T & -u_1P_1^T \\ 0^T & P_1^T & -v_1P_1^T \\ \cdots & \cdots & \cdots \\ P_n^T & 0^T & -u_nP_n^T \\ 0^T & P_n^T & -v_nP_n^T \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix} = \begin{pmatrix} P_1^T\lambda - u_1P_1^T\nu \\ P_1^T\mu - v_1P_1^T\nu \\ \cdots \\ P_n^T\lambda - u_nP_n^T\nu \\ P_n^T\mu - v_nP_n^T\nu \end{pmatrix}$$

- Coplanar points: $(\lambda, \mu, \nu) = (\Pi, 0, 0)$ or $(0, \Pi, 0)$ or $(0, 0, \Pi)$
- Points lying on the intersection curve of two quadric surfaces = straight line + twisted cubic

Does not happen for 6 or more random points!
Analytical Photogrammetry

Given $n$ points $P_1, \ldots, P_n$ with known positions and their images $p_1, \ldots, p_n$

Find $i$ and $e$ such that

$$\sum_{i=1}^{n} \left[ \left( u_i - \frac{m_1(i, e) \cdot P_i}{m_3(i, e) \cdot P_i} \right)^2 + \left( v_i - \frac{m_2(i, e) \cdot P_i}{m_3(i, e) \cdot P_i} \right)^2 \right]$$

is minimized

Non-Linear Least-Squares Methods

- Newton
- Gauss-Newton
- Levenberg-Marquardt

Iterative, quadratically convergent in favorable situations
Mobile Robot Localization (Devy et al., 1997)