Heat Equation (Diffusion) [Pao79] [EK88] [SR66]

\( q = \text{rate of heat flow per unit area} \)
\( u = \text{temperature} \)

\[ \frac{\partial q}{\partial x} = \frac{\partial q}{\partial (x + \Delta x, t)} + \text{rate of heat storage} \]

\[ \text{rate of heat storage} \sim \text{rate of change of temperature} u \]

\[ \Delta x \rho C \frac{\partial u}{\partial t} \]

\[ \text{Heat capacity per unit mass} \]

\[ \frac{q(x + \Delta x, t) - q(x, t)}{\Delta x} = -\rho C \frac{\partial u}{\partial t} \]

Limit as \( \Delta x \to 0 \)

Independent of \( \Delta x \)

\[ \frac{\partial^2 q}{\partial x^2} = -\rho C \frac{\partial u}{\partial t} \]

Rate of heat flow \( \sim \) temperature gradient

Fourier's Law: "heat flows downhill," or \( q = -K \frac{\partial u}{\partial x} \)

Heat Eqn.

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\rho C}{k} \frac{\partial u}{\partial t} \]
Wave Equation

Consider stretched string between two fixed points:

\[ y \] \[ x \] Vibrates

\[ \psi_{\text{left}} \] \[ \psi_{\text{right}} \]

blow up tiny piece

\[ \psi_{\text{left}} \] \[ \psi_{\text{right}} \]

but vertical component of force on segment

\[ P \sin \psi_{\text{right}} - P \sin \psi_{\text{left}} = ma = \frac{P}{\Delta x} \frac{\partial^2 y}{\partial t^2} \]

\[ \frac{\partial^2 y}{\partial t^2} = \frac{P}{s} \left[ \frac{\sin \psi_{\text{right}} - \sin \psi_{\text{left}}}{\Delta x} \right] \]

In reality, \( \psi \) is very small, so

\[ \sin \psi \approx \tan \psi \approx \frac{\partial \psi}{\partial x} \]

\[ \frac{\partial^2 y}{\partial t^2} = \frac{P}{s} \left[ \frac{\partial^2 y}{\partial x^2} \right. \left. - \frac{\partial^2 y}{\partial x} \right] \]

Limit as \( \Delta x \to 0 \)

\[ \frac{\partial^2 y}{\partial t^2} = \frac{P}{s} \frac{\partial^2 y}{\partial x^2} \]

Wave Eqn.

Usually denoted \( c^2 \), dimensions of velocity^2
we'll concentrate on wave eqns.

\[
\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}
\]

Guess a bump moving to the right on the string

\[ y \]

\[ x \]

this is of the form \( f(t-x/c) \)

speed is \( c \): increase \( t+\Delta t \)

\[
( t + \Delta t - \frac{x + c \Delta t}{c} ) = t - \frac{x}{c} \checkmark
\]

\[ x = x + c \Delta t \]

\[ \rightarrow \Delta x/\Delta t = c \]

Check Eqn:

\[
\frac{\partial}{\partial t} = f''(t-x/c)
\]

\[
c^2 \frac{\partial^2}{\partial x^2} = \frac{1}{c^2} \frac{\partial}{\partial t} \checkmark
\]

this is true no matter what the shape \( f(\cdot) \)

Same argument works for bump moving left

\[ g(t+x/c) \]

If any two solns. are added, the sum also satisfies wave eqn. So we always have as a soln.

\[
f(t-x/c) + g(t+x/c)
\]

\[ \text{general form} \]

right-moving wave

left-moving wave

what causes periodic vibrations, like guitar string?
Examples:

→ ←

→ ←

→ ←

→ ←

→ ←

→ ←

→ ←

→ ←

String is flat!
Boundary Conditions: String is tied down at ends, say: 5

\[ y(0,t) = f(t) + g(t) = 0, \quad \text{all } t \]
\[ \Rightarrow f(t) = -g(t) \]

So general solution becomes

\[ y(x,t) = f(t - x/c) - f(t + x/c) \]

Interpretation:

And then

\[ \therefore \] wave shape is inverted on reflection from fixed end.
Second fixed point:

\[ y(L, t) = f(t - \frac{L}{c}) - f(t + \frac{L}{c}) = 0 \]

In other words, \( f(t) = f(t + \frac{2L}{c}) \) \text{ periodic in } t

What is \( \frac{2L}{c} \)? \text{ Roundtrip time}

We're going to look for solutions with this period

\[ f_0 = \frac{1}{T_0} = \frac{c}{2L} \text{ Hz (cycles/sec)} \]

\[ \omega_0 = 2\pi f_0 = \frac{2\pi c}{L} \text{ radians/sec} \]

\[ T_0 = \frac{2L}{c} \text{ sec} \]

Also, instead of \( \sin(\omega_0 t) \) \& \( \cos(\omega_0 t) \), we'll use

\[ e^{j\omega t} = \cos(\omega t) + j \sin(\omega t) \text{ complex exponential} \]

This form greatly simplifies algebra, and will be useful later for (1) stability analysis — and still later for Fourier analysis.
We guess a solution of the form

\[ y(x, t) = e^{j\omega t} Y(x) \]

plug into wave equation,

\[ \frac{\partial^2 y}{\partial t^2} = -\omega_0^2 e^{j\omega t} Y(x) \]

\[ c^2 \frac{\partial^2 y}{\partial x^2} = c^2 e^{j\omega t} Y''(x) \]

\[ y''(x) = -\frac{\omega_0^2}{c^2} y(x) = -\frac{\pi^2 x^2}{L^2} c^2 y(x) \]

(this is an ODE)

Solution:

\[ Y(x) = \sin (\frac{\pi x}{L} + \phi) \]

We have yet to determine \( \phi \), so use boundary conditions

\[ Y(x) = 0 \text{ at } x = 0 \text{ and } L. \]

Yields

\[ \sin \phi = \sin (\pi + \phi) = 0 \]

\[ \Rightarrow \phi = 0, \pm \pi, \pm 2\pi, \ldots \]

Doesn't matter, use \( \phi = 0 \), so solution is

\[ y(x, t) = e^{j\omega t} \sin (\frac{\pi x}{L}) \]

(If you want, take real part, since this is complex.)
But, we could have also tried
\[ e^{i(2\omega t)}Y(x) \]
\[ e^{i(\omega t)}Y(x) \ldots \text{etc.} \]
These lead to solutions:
\[ e^{i\omega t} \sin(\frac{k\pi x}{L}) \]
Sums of solutions are also solutions (linear eqn.)
So general soln. is
\[
\sum_{k=1}^{\infty} C_k e^{i\omega t} \sin\left(\frac{k\pi x}{L}\right)
\]
Interpretation:
- Frequency $\omega_0$
- Mode 1
- Frequency $2\omega_0$
- Mode 2
- Frequency $3\omega_0$
- Mode 3
\[ \vdots \]
\[
\sum_{k=-\infty}^{\infty} C_k e^{jk\omega t} \sin(k\pi x/L)
\]

can be any initial shape \( \Rightarrow \) Fourier series can represent any function.

Back to general solution, let \( c=1 \) for now:

\[
y(x,t) = f(x+ct) + g(x-ct)
\]

Usually, we are given \( y(x,0) \) & \( y'(x,0) \)

Differentiate:

\[
\begin{align*}
\begin{cases}
\frac{df}{dx} + \frac{dg}{dx} = F(x) & \text{given initial shape} \\
\frac{df}{dx} - \frac{dg}{dx} = G(x) & \text{given initial velocity} \\
\frac{df}{dx} + \frac{dg}{dx} = F'(x)
\end{cases}
\]

Add & subtract,

\[
\begin{align*}
\begin{cases}
\frac{d}{dx}f(x) &= \frac{1}{2} \left[ F'(x) + G(x) \right] \\
\frac{d}{dx}g(x) &= \frac{1}{2} \left[ F'(x) - G(x) \right]
\end{cases}
\]

\[
\begin{align*}
\begin{cases}
f(x) = \frac{1}{2} \left[ F(x) + \int_{0}^{x} G(y)dy \right] + C \\
g(x) = \frac{1}{2} \left[ F(x) - \int_{0}^{x} G(y)dy \right] + D
\end{cases}
\]

& use:

\[
y(x,t) = f(x+t) + g(x-t)
\]
\[ y(x,t) = \frac{1}{2} \left[ F(x+t) + F(x-t) + \int_{x-t}^{x+t} G(\eta) \, d\eta \right] + \frac{1}{2} \int_{-\infty}^{\infty} F(x) \, dx \] for general \( c \), this

\textit{d'Alembert's formula}

\[ y(x,t) = \frac{1}{2} F(x+ct) + \frac{1}{2} F(x-ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} G(\eta) \, d\eta \]

at time \( t^* \), what values of \( F \) & \( G \) affect soln?

only soln. here \( t^* \) can affect solution at \((x,t^*)\)

this form can be used for calculation, but gets too complicated in finite string with reflections.