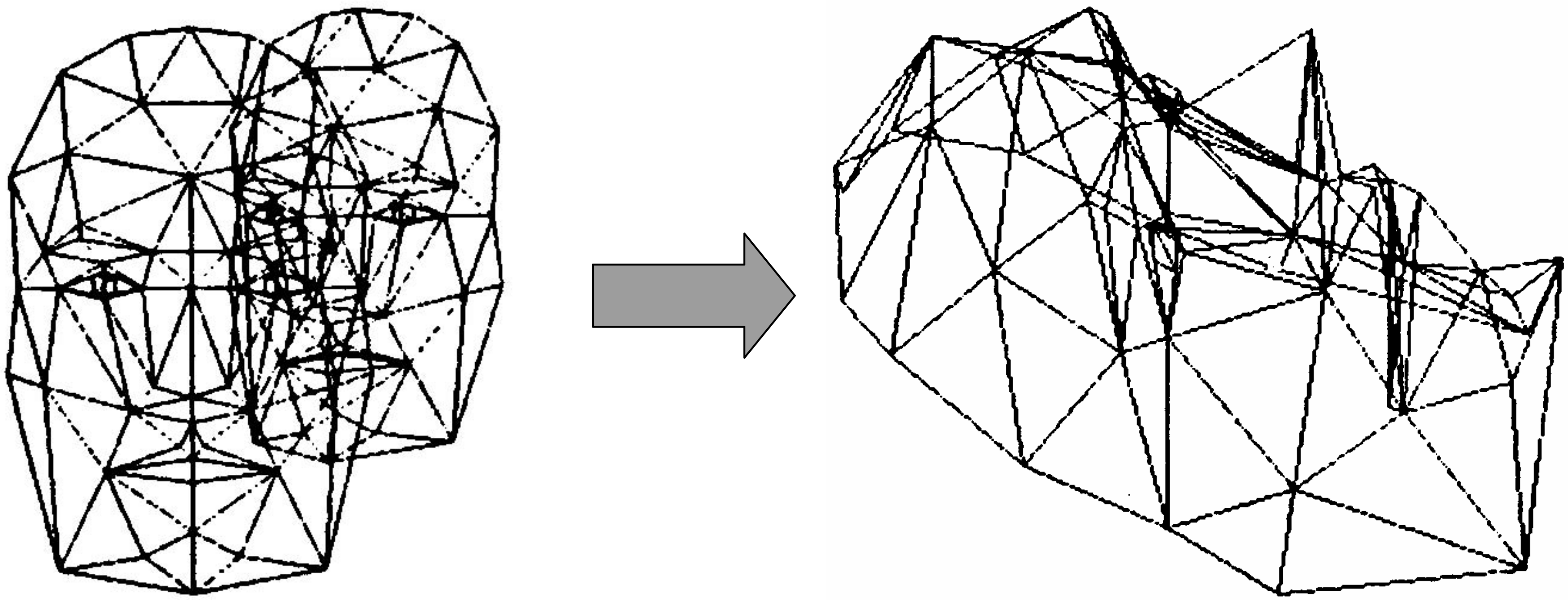
COS429: COMPUTER VISON
AFFINE STRUCTURE FROM MOTION

The Structure-from-Motion Problem
Affine Projection Models
Affine Ambiguity of Affine SFM
Affine Epipolar Geometry
Affine Reconstruction from two Images
Affine Reconstruction from Multiple Images

- Reading: Chapter 12

Many of the slides in this lecture are courtesy to Prof. J. Ponce

Affine Structure from Motion



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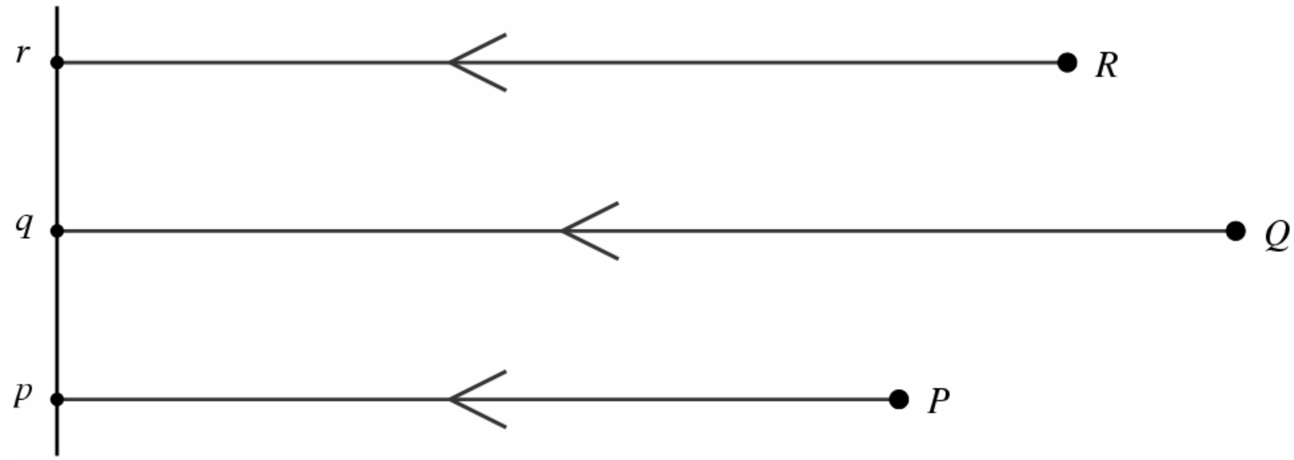
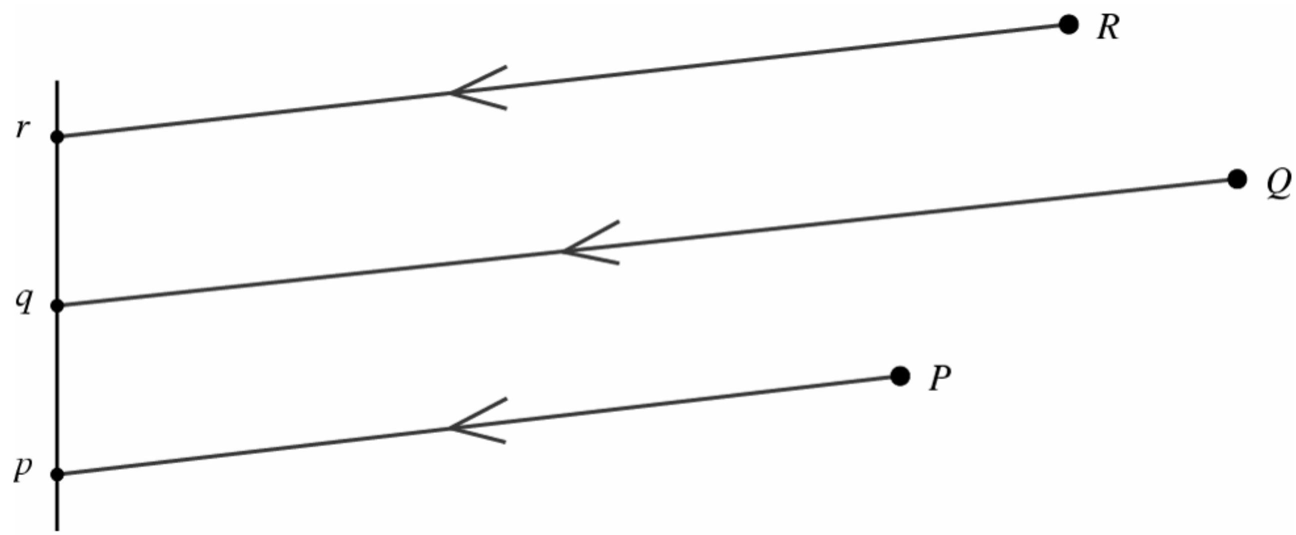
Given *m* pictures of *n* points, can we recover
- the three-dimensional configuration of these points? (structure)
- the camera configurations? (motion)

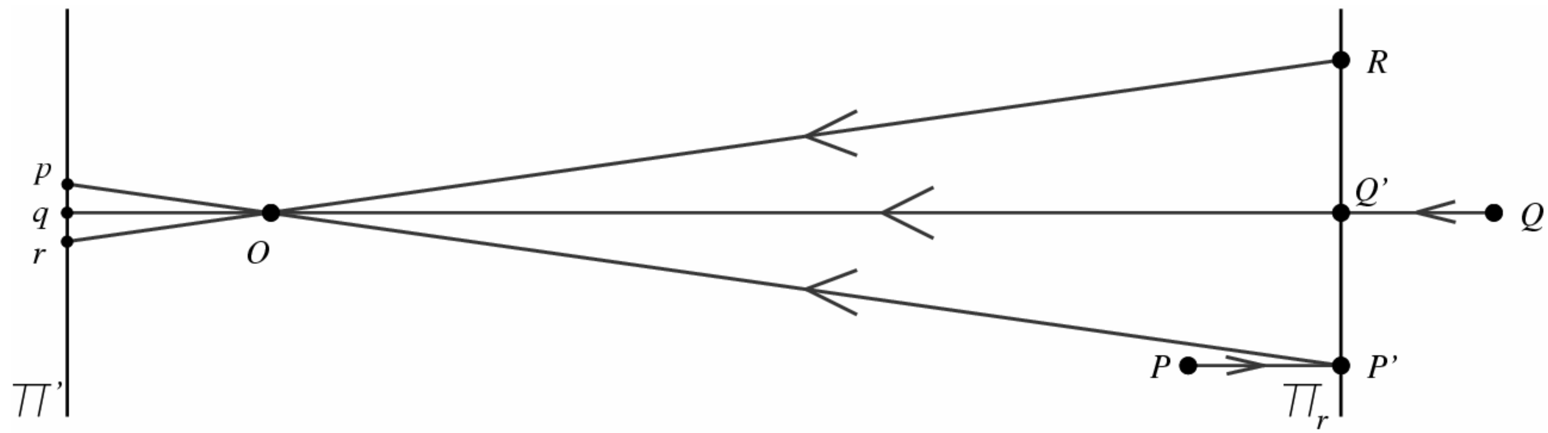
Orthographic Projection



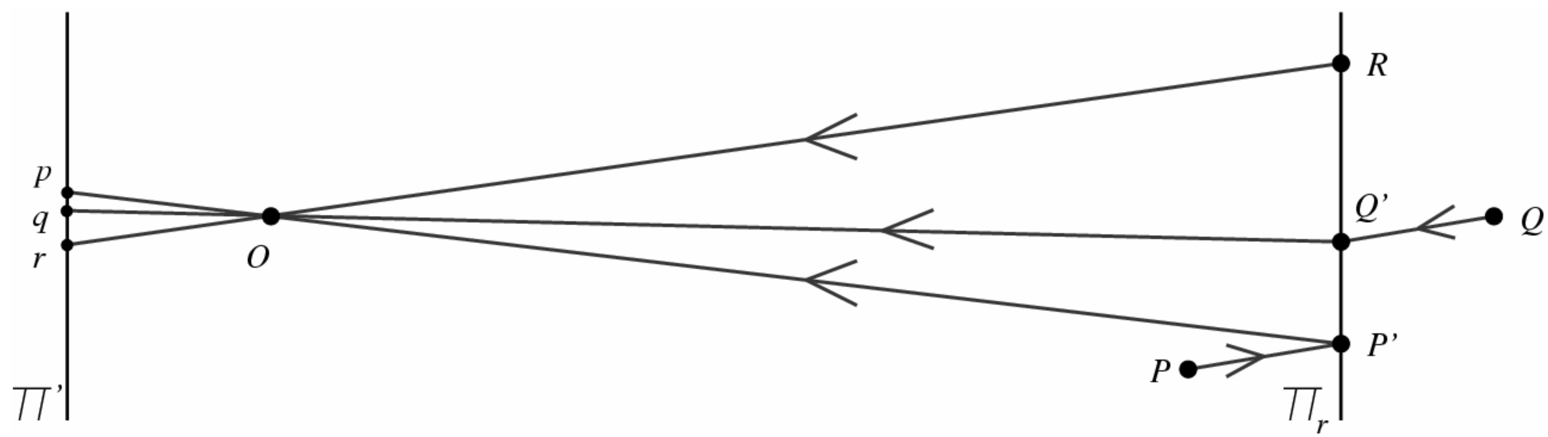
Parallel Projection



Weak-Perspective Projection



Paraperspective Projection



The Affine Structure-from-Motion Problem

Given m images of n fixed points P_j we can write

$$\boldsymbol{p}_{ij} = \mathcal{M}_i \begin{pmatrix} \boldsymbol{P}_j \\ 1 \end{pmatrix} = \mathcal{A}_i \boldsymbol{P}_j + \boldsymbol{b}_i \quad \text{for} \quad i = 1, \ldots, m \quad \text{and} \quad j = 1, \ldots, n.$$

Problem: estimate the m 2x4 matrices $\mathcal{M}_i$ and
the n positions P_j from the mn correspondences $\boldsymbol{p}_{ij}$.

$2mn$ equations in $8m+3n$ unknowns

Overconstrained problem, that can be solved
using (non-linear) least squares!

The Affine Epipolar Constraint

$$\begin{cases} \boldsymbol{p} = \mathcal{A}\boldsymbol{P} + \boldsymbol{b} \\ \boldsymbol{p}' = \mathcal{A}'\boldsymbol{P} + \boldsymbol{b}' \end{cases} \Rightarrow \begin{pmatrix} \mathcal{A} & \boldsymbol{p} - \boldsymbol{b} \\ \mathcal{A}' & \boldsymbol{p}' - \boldsymbol{b}' \end{pmatrix} \begin{pmatrix} \boldsymbol{P} \\ -1 \end{pmatrix} = \boldsymbol{0}$$

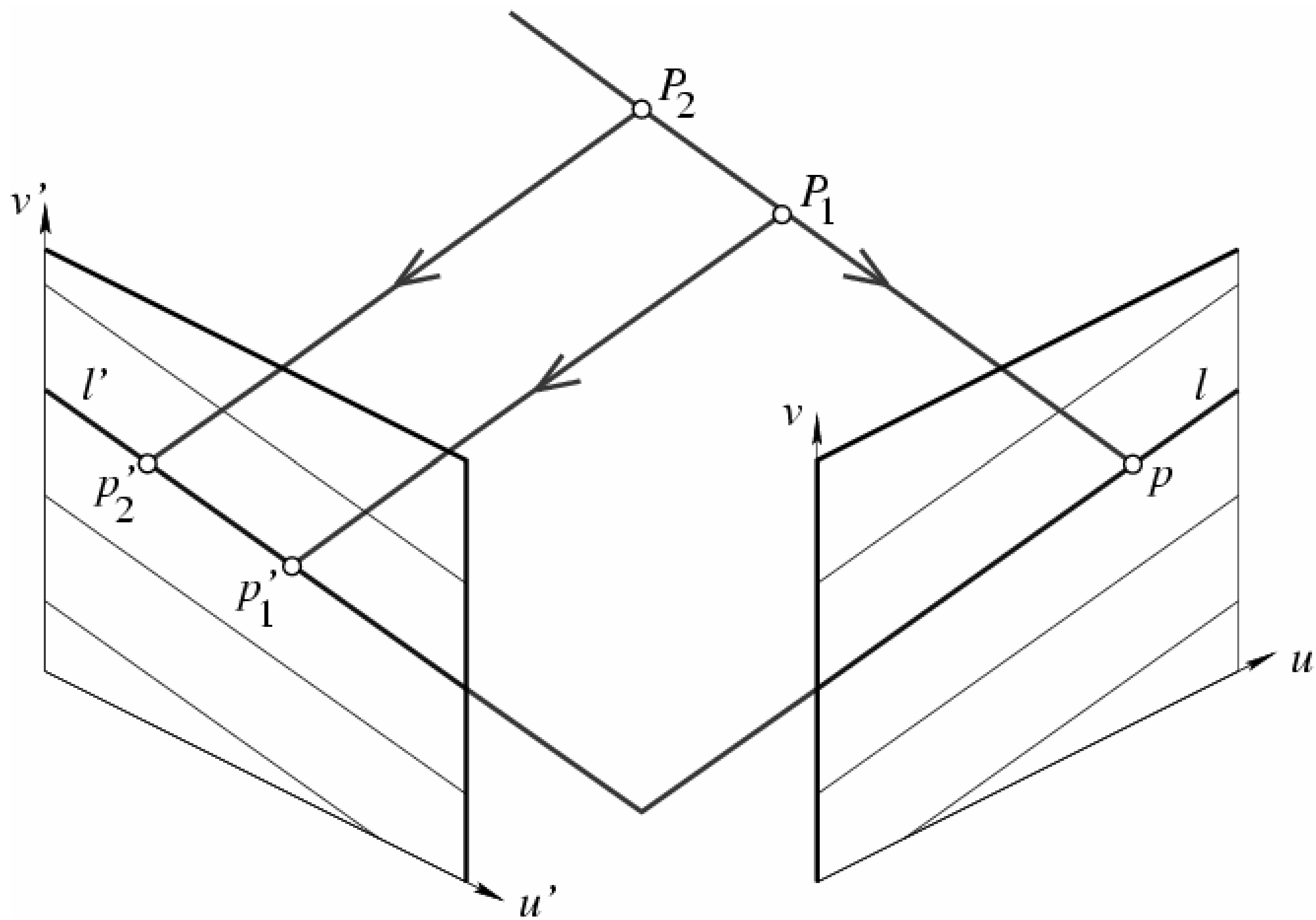
$$\Downarrow$$

$$\operatorname{Det} \begin{pmatrix} \mathcal{A} & \boldsymbol{p} - \boldsymbol{b} \\ \mathcal{A}' & \boldsymbol{p}' - \boldsymbol{b}' \end{pmatrix} = 0$$

$$\Downarrow$$

$$\boxed{\alpha u + \beta v + \alpha' u' + \beta' v' + \delta = 0}$$

Affine Epipolar Geometry



Note: the epipolar lines are parallel.

The Affine Fundamental Matrix

$$\alpha u + \beta v + \alpha' u' + \beta' v' + \delta = 0$$

$$\Downarrow$$

$$(u, v, 1)\mathcal{F}\begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0 \qquad \text{where} \qquad \mathcal{F} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \\ \alpha' & \beta' & \delta \end{pmatrix}$$

The Affine Ambiguity of Affine SFM

When the intrinsic and extrinsic parameters are unknown

If M_i and P_j are solutions,

$$\boldsymbol{p}_{ij} = \mathcal{M}_i \begin{pmatrix} \boldsymbol{P}_j \\ 1 \end{pmatrix} = (\mathcal{M}_i \mathcal{Q}) \left(\mathcal{Q}^{-1} \begin{pmatrix} \boldsymbol{P}_j \\ 1 \end{pmatrix}\right) = \mathcal{M}'_i \begin{pmatrix} \boldsymbol{P}'_j \\ 1 \end{pmatrix}$$

So are M'_i and P'_j where

$$\mathcal{M}'_i = \mathcal{M}_i \mathcal{Q} \quad \text{and} \quad \begin{pmatrix} \boldsymbol{P}'_j \\ 1 \end{pmatrix} = \mathcal{Q}^{-1} \begin{pmatrix} \boldsymbol{P}_j \\ 1 \end{pmatrix}$$

and

$$\mathcal{Q} = \begin{pmatrix} \mathcal{C} & \boldsymbol{d} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \quad \text{with} \quad \mathcal{Q}^{-1} = \begin{pmatrix} \mathcal{C}^{-1} & -\mathcal{C}^{-1}\boldsymbol{d} \\ \boldsymbol{0}^T & 1 \end{pmatrix}$$

Q is an affine transformation.

An Affine Trick.. Algebraic Scene Reconstruction Method

$$\mathcal{M} = (\mathcal{A} \quad \boldsymbol{b}) \qquad \mathcal{M}' = (\mathcal{A}' \quad \boldsymbol{b}') \qquad \boldsymbol{P}$$

$$\tilde{\mathcal{M}} = \mathcal{M}\mathcal{Q} \qquad \tilde{\mathcal{M}}' = \mathcal{M}'\mathcal{Q} \qquad \tilde{\boldsymbol{P}} = \mathcal{Q}^{-1}\boldsymbol{P}$$

$$\tilde{\mathcal{M}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \tilde{\mathcal{M}}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ a & b & c & d \end{pmatrix} \qquad \tilde{\boldsymbol{P}}$$

$$\boxed{\mathrm{Det}\begin{pmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & u' \\ a & b & c & v'-d \end{pmatrix} = au - bv + cu' + v' - d = 0}$$

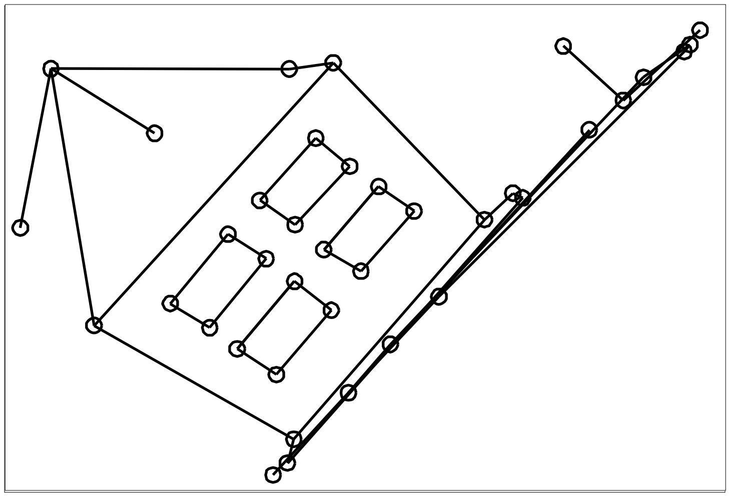
An Affine Trick.. $\Rightarrow$ Algebraic Scene Reconstruction Method

$$\mathcal{M} = \begin{pmatrix} \mathcal{A} & \boldsymbol{b} \end{pmatrix} \qquad \mathcal{M}' = \begin{pmatrix} \mathcal{A}' & \boldsymbol{b}' \end{pmatrix} \qquad \boldsymbol{P}$$

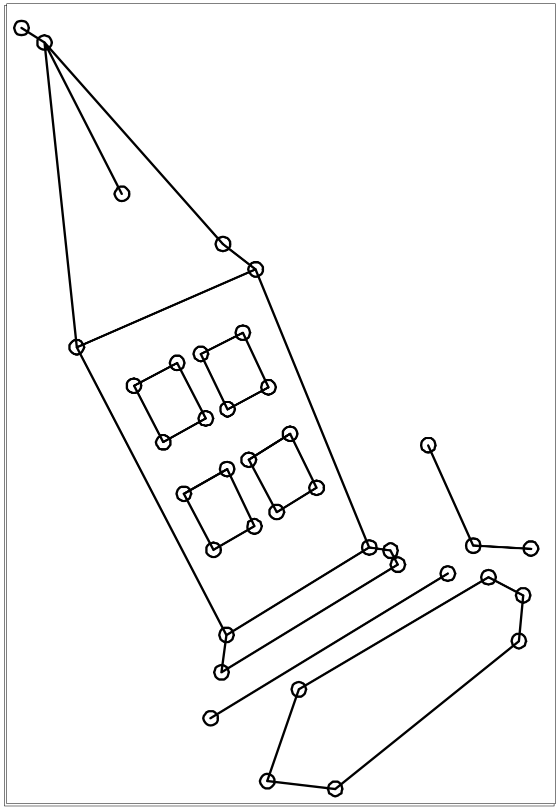
$$\tilde{\mathcal{M}} = \mathcal{M}\mathcal{Q} \qquad \tilde{\mathcal{M}}' = \mathcal{M}'\mathcal{Q} \qquad \tilde{\boldsymbol{P}} = \mathcal{Q}^{-1}\boldsymbol{P}$$

$$\tilde{\mathcal{M}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \tilde{\mathcal{M}}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ a & b & c & d \end{pmatrix} \quad \tilde{\boldsymbol{P}}$$

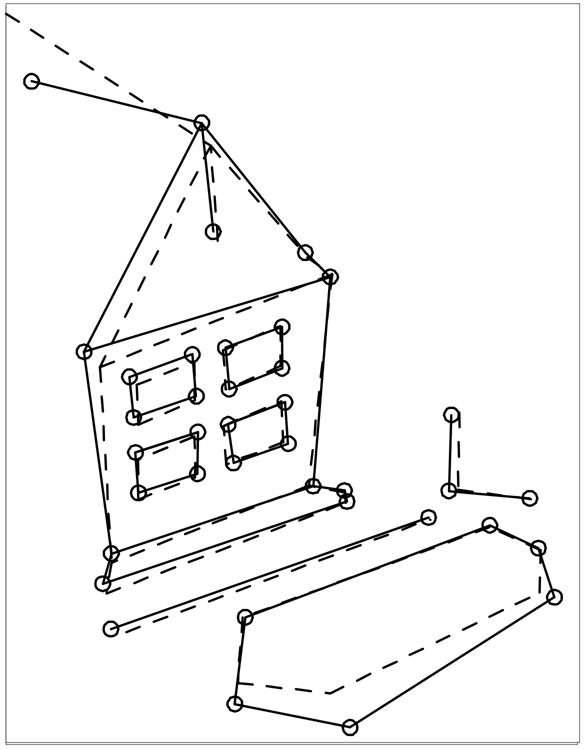
$$\begin{pmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & u' \\ a & b & c & v' - d \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{P}} \\ -1 \end{pmatrix} = 0 \quad \Rightarrow \quad \boxed{\tilde{\boldsymbol{P}} = \begin{pmatrix} u \\ v \\ u' \end{pmatrix}}$$



First reconsruction. Mean reprojection error: 1.6pixel



Second reconsruction. Mean reprojection error: 7.8pixel



Suppose we observe a scene with *m* fixed cameras..

$$\boldsymbol{p}_i = \mathcal{M}_i \begin{pmatrix} \boldsymbol{P} \\ 1 \end{pmatrix} = \mathcal{A}_i \boldsymbol{P} + \boldsymbol{b}_i \quad \text{for} \quad i = 1, \ldots, m$$

$$\begin{aligned} \boldsymbol{P} &\longrightarrow \boldsymbol{P} - \boldsymbol{P}_0 \\ \boldsymbol{p} &\longrightarrow \boldsymbol{p} - \boldsymbol{p}_0 \end{aligned} \quad \Longrightarrow \quad \boldsymbol{p}_i = \mathcal{A}_i \boldsymbol{P} + \boldsymbol{b}_i \longrightarrow \boldsymbol{p}_i = \mathcal{A}_i \boldsymbol{P}$$

$$\boxed{\begin{matrix} u_{11}\ u_{12} \ldots u_{1n} \\ v_{11}\ v_{12} \ldots v_{1n} \\ \ldots\ldots\ldots \\ u_{m1}\ u_{m2} \ldots u_{mn} \\ v_{m1}\ v_{m2} \ldots v_{mn} \end{matrix}} = \boxed{\begin{matrix} A_1 \\ A_2 \\ \ldots \\ A_m \end{matrix}} \boxed{P_1\ P_2 \ldots P_n} = \mathcal{D}$$

What if we could factorize D? (Tomasi and Kanade, 1992)

$$\mathcal{A}, \mathcal{P} \rightarrow \mathcal{D}$$

Affine SFM is solved!

$$\mathcal{D} \rightarrow \mathcal{A}, \mathcal{P}$$

Singular Value Decomposition

Let $\mathcal{A}$ be an $m \times n$ matrix, with $m \geq n$, then $\mathcal{A}$ can always be written as

$$\mathcal{A} = \mathcal{U}\mathcal{W}\mathcal{V}^T,$$

where:

- $\mathcal{U}$ is an $m \times n$ column-orthogonal matrix, i.e., $\mathcal{U}^T\mathcal{U} = \mathrm{Id}_m$,
- $\mathcal{W}$ is a diagonal matrix whose diagonal entries w_i $(i = 1, \ldots, n)$ are the singular values of $\mathcal{A}$ with $w_1 \geq w_2 \geq \ldots \geq w_n \geq 0$,
- and $\mathcal{V}$ is an $n \times n$ orthogonal matrix, i.e., $\mathcal{V}^T\mathcal{V} = \mathcal{V}\mathcal{V}^T = \mathrm{Id}_n$.

What if we could factorize D? (Tomasi and Kanade, 1992)

$$\mathcal{A}, \mathcal{P} \rightarrow \mathcal{D}$$

Affine SFM is solved!

$$\mathcal{D} \rightarrow \mathcal{A}, \mathcal{P}$$

Singular Value Decomposition

Theorem: The singular values of the matrix $\mathcal{A}$ are the eigenvalues of the matrix $\mathcal{A}^T\mathcal{A}$ and the columns of the matrix $\mathcal{V}$ are the corresponding eigenvectors.

What if we could factorize D? (Tomasi and Kanade, 1992)

$\mathcal{A}, \mathcal{P} \to \mathcal{D}$

Affine SFM is solved!

$\mathcal{D} \to \mathcal{A}, \mathcal{P}$

Singular Value Decomposition

When $\mathcal{A}$ has rank $p < n$, then the matrices $\mathcal{U}$, $\mathcal{W}$, and $\mathcal{V}$ can be written as

$$\mathcal{U} = \begin{array}{|c|c|} \hline \mathcal{U}_p & \mathcal{U}_{n-p} \\ \hline \end{array} \quad \mathcal{W} = \begin{array}{|c|c|} \hline \mathcal{W}_p & 0 \\ \hline 0 & 0 \\ \hline \end{array} \quad \text{and} \quad \mathcal{V}^T = \begin{array}{|c|} \hline \mathcal{V}_p^T \\ \hline \mathcal{V}_{n-p}^T \\ \hline \end{array},$$

and

- the columns of $\mathcal{U}_p$ form an orthonormal basis of the space spanned by the columns of $\mathcal{A}$, i.e., its *range*,
- and the columns of $\mathcal{V}_{n-p}$ for a basis of the space spanned by the solutions of $A\boldsymbol{x} = 0$, i.e., the *null space* of this matrix.

In addition, $\mathcal{A} = \mathcal{U}_p \mathcal{W}_p \mathcal{V}_p^T$.

What if we could factorize D? (Tomasi and Kanade, 1992)

$$\mathcal{A}, \mathcal{P} \rightarrow \mathcal{D}$$

Affine SFM is solved!

$$\mathcal{D} \rightarrow \mathcal{A}, \mathcal{P}$$

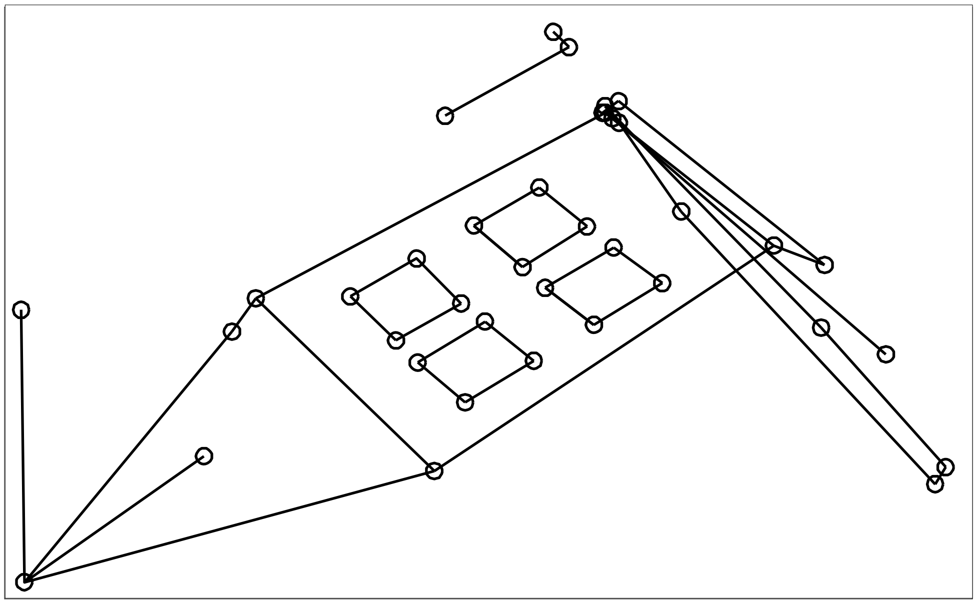
$$E \stackrel{\text{def}}{=} \sum_{i,j} |\boldsymbol{p}_{ij} - \mathcal{A}_i \boldsymbol{P}_j|^2 = \sum_j |\boldsymbol{q}_j - \mathcal{A}\boldsymbol{P}_j|^2 = |\mathcal{D} - \mathcal{A}\mathcal{P}|^2$$

Singular Value Decomposition

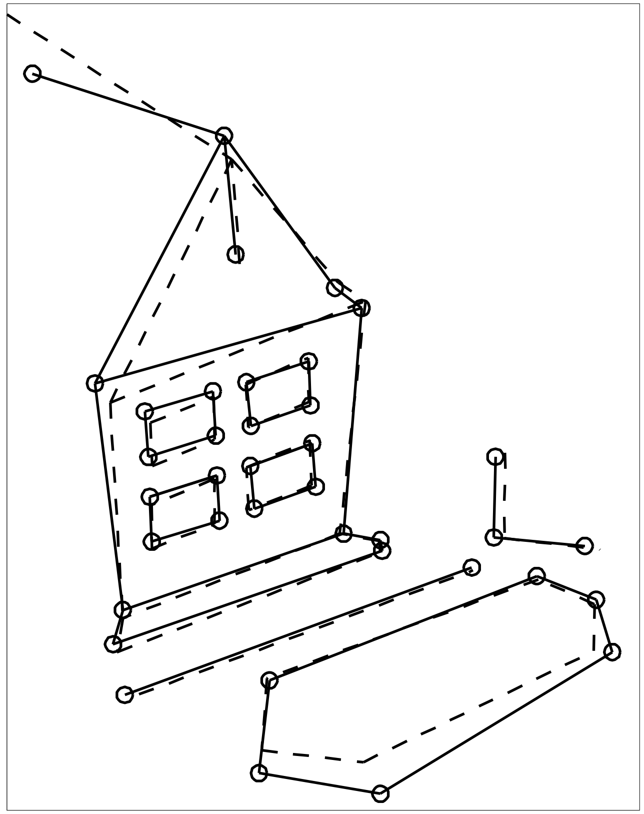
Theorem: When $\mathcal{A}$ has a rank greater than p, $\mathcal{U}_p \mathcal{W}_p \mathcal{V}_p^T$ is the best possible rank-p approximation of $\mathcal{A}$ in the sense of the Frobenius norm.

$$\mathcal{D} = \mathcal{U}_3 \mathcal{W}_3 \mathcal{V}_3^T$$

$$\boxed{\begin{cases} \mathcal{A}_0 = \mathcal{U}_3 \\ \mathcal{P}_0 = \mathcal{W}_3 \mathcal{V}_3^T \end{cases}}$$



Mean reprojection error:
2.4pixel



From uncalibrated to calibrated cameras

Weak-perspective camera:

$$\mathcal{M} = \frac{1}{z_r}\begin{pmatrix} k & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathcal{R}_2 & \boldsymbol{t}_2 \end{pmatrix}$$

$$\hat{\mathcal{M}} = \mathcal{M}\mathcal{Q}$$

Calibrated camera:

$$\hat{\mathcal{M}} = \begin{pmatrix} \hat{\mathcal{A}} & \hat{\boldsymbol{b}} \end{pmatrix} = \frac{1}{z_r}\begin{pmatrix} \mathcal{R}_2 & \boldsymbol{t}_2 \end{pmatrix}$$

Problem: what is *Q* ?

$$\hat{\boldsymbol{a}}_1 \cdot \hat{\boldsymbol{a}}_2 = 0 \quad \text{and} \quad |\hat{\boldsymbol{a}}_1|^2 = |\hat{\boldsymbol{a}}_2|^2$$

$$\mathcal{Q} = \begin{pmatrix} \mathcal{C} & \boldsymbol{d} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \Rightarrow \begin{cases} \boldsymbol{a}_{i1}^T \mathcal{C}\mathcal{C}^T \boldsymbol{a}_{i2} = 0, \\ \boldsymbol{a}_{i1}^T \mathcal{C}\mathcal{C}^T \boldsymbol{a}_{i1} = 1, \\ \boldsymbol{a}_{i2}^T \mathcal{C}\mathcal{C}^T \boldsymbol{a}_{i2} = 1, \end{cases} \quad \text{for} \quad i = 1, \ldots, m,$$

From uncalibrated to calibrated cameras

Weak-perspective camera:

$$\mathcal{M} = \frac{1}{z_r}\begin{pmatrix} k & s \\ 0 & 1 \end{pmatrix}(\,\mathcal{R}_2 \quad \boldsymbol{t}_2\,)$$

$$\hat{\mathcal{M}} = \mathcal{M}\mathcal{Q}$$

Calibrated camera:

$$\hat{\mathcal{M}} = (\,\hat{\mathcal{A}} \quad \hat{\boldsymbol{b}}\,) = \frac{1}{z_r}(\,\mathcal{R}_2 \quad \boldsymbol{t}_2\,)$$

Problem: what is *Q* ?

$$\hat{\boldsymbol{a}}_1 \cdot \hat{\boldsymbol{a}}_2 = 0 \quad \text{and} \quad |\hat{\boldsymbol{a}}_1|^2 = |\hat{\boldsymbol{a}}_2|^2$$

$$\mathcal{D} = \mathcal{C}\mathcal{C}^T \Rightarrow \begin{cases} \boldsymbol{a}_{i1}^T \mathcal{D} \boldsymbol{a}_{i2} = 0, \\ \boldsymbol{a}_{i1}^T \mathcal{D} \boldsymbol{a}_{i1} = \boldsymbol{a}_{i2}^T \mathcal{D} \boldsymbol{a}_{i2} \end{cases} \quad \text{for} \quad i = 1, \ldots, m.$$

From uncalibrated to calibrated cameras

Weak-perspective camera: $$\mathcal{M} = \frac{1}{z_r}\begin{pmatrix} k & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathcal{R}_2 & \boldsymbol{t}_2 \end{pmatrix}$$

$$\hat{\mathcal{M}} = \mathcal{M}\mathcal{Q}$$

Calibrated camera: $$\hat{\mathcal{M}} = \begin{pmatrix} \hat{\mathcal{A}} & \hat{\boldsymbol{b}} \end{pmatrix} = \frac{1}{z_r}\begin{pmatrix} \mathcal{R}_2 & \boldsymbol{t}_2 \end{pmatrix}$$

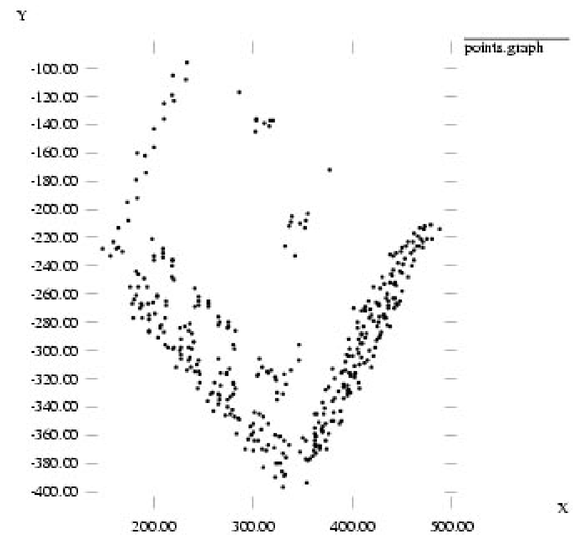
Problem: what is *Q* ? $$\hat{\boldsymbol{a}}_1 \cdot \hat{\boldsymbol{a}}_2 = 0 \quad \text{and} \quad |\hat{\boldsymbol{a}}_1|^2 = |\hat{\boldsymbol{a}}_2|^2$$

Note: Absolute scale cannot be recovered. The Euclidean shape (defined up to an arbitrary similitude) is recovered.

Reconstruction Results (Tomasi and Kanade, 1992)

1

60



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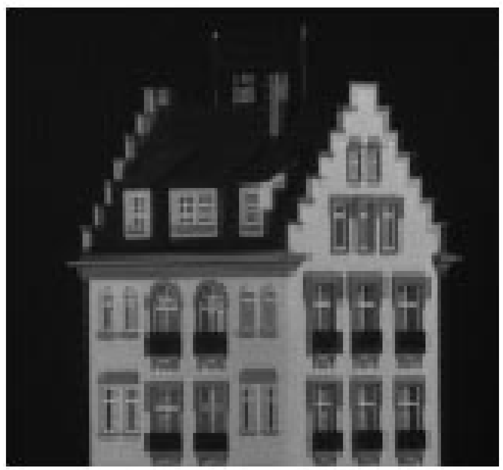

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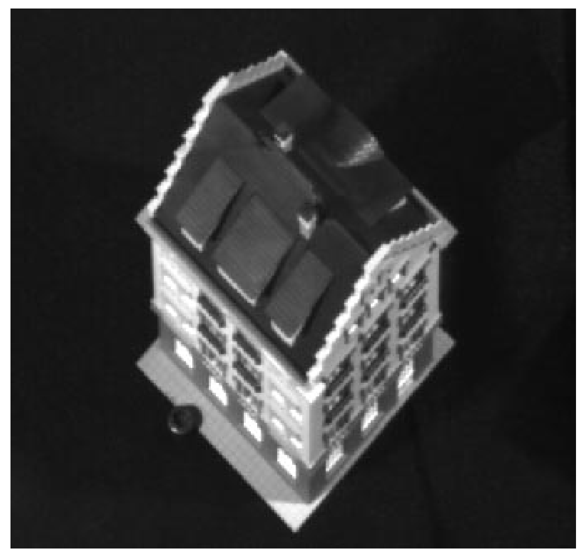
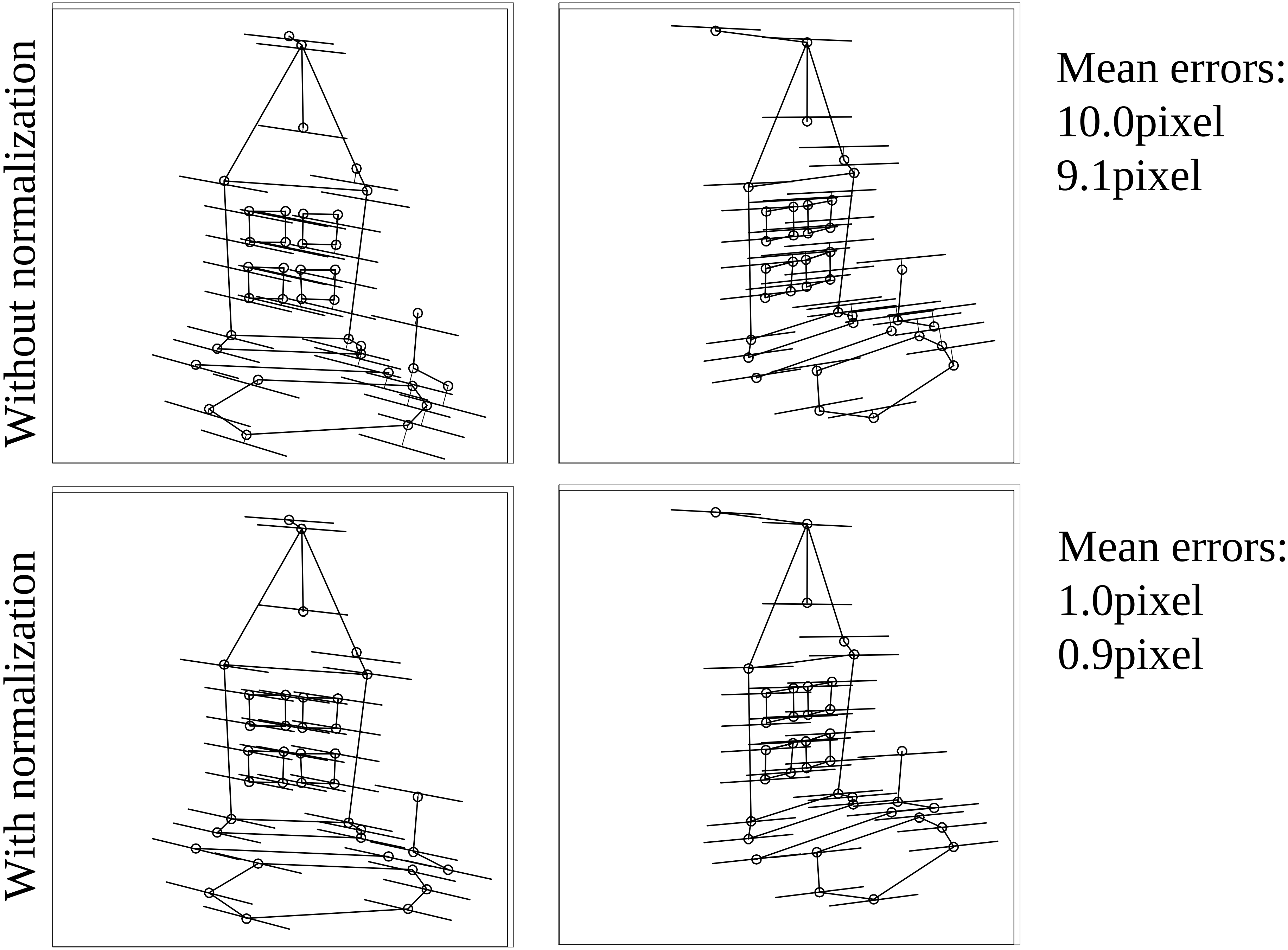
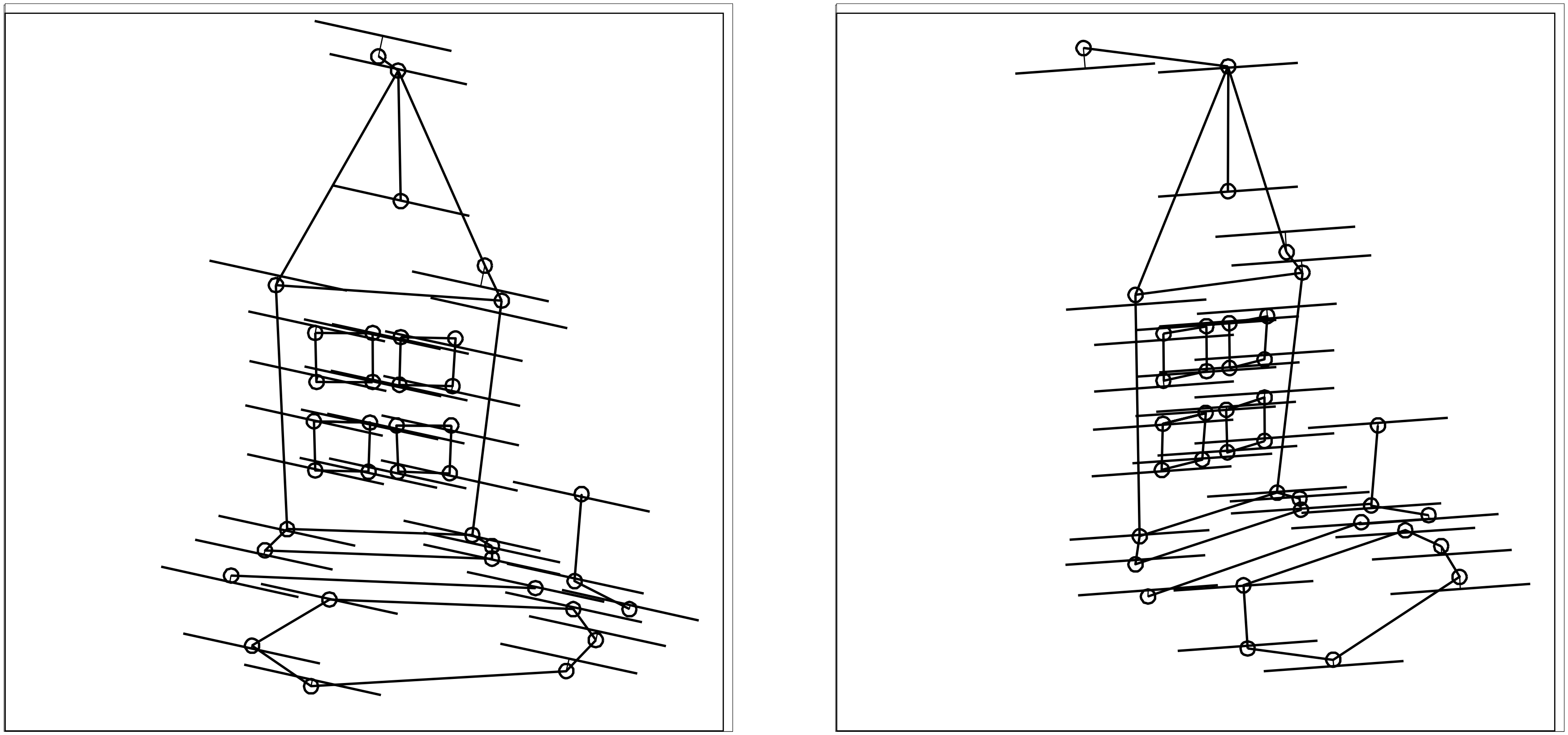


Reprinted from "Factoring Image Sequences into Shape and Motion," by C. Tomasi and T. Kanade, Proc. IEEE Workshop on Visual Motion (1991). © 1991 IEEE.


Without normalization
Mean errors:
10.0pixel
9.1pixel
With normalization
Mean errors:
1.0pixel
0.9pixel



Mean errors: 3.24 and 3.15 pixels