Nonlinear Vibrating String

Sonification

When simulating a physical system like a vibrating string, which makes sound waves in the real world, it makes sense to *listen* to the results, rather than look at plots. The ear is in many (but not all) respects a very good spectrum analyzer, and you will get more insight (inhearing?) by using your well-tuned ears. The mechanics of getting playable sound files are relatively straightforward. Some utility programs in C are supplied in the sonification lecture notes (around the beginning of November); or you can use Matlab, which has built-in functions for input and output of audio files.

Basic Part: Ideal String

If $y(x,t)$ is the displacement of a string stretched between two points, and if the displacement is small, vibration is determined by the wave equation:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

The assignment is to simulate a vibrating string using a finite-difference method, starting from various initial conditions. To do this, first discretize time and space by

$$y(m\Delta x, n\Delta t) = y_{mn} = y(x,t)$$

Then use the following central-difference approximation for $\partial^2 y/\partial t^2$ [1, pp. 213*ff*]:

$$\frac{\partial^2 y}{\partial t^2} \approx \frac{y_{m,n-1} - 2y_{mn} + y_{m,n+1}}{(\Delta t)^2}$$

and the analogous approximation for the right-hand side of the wave equation. Then derive an explicit iteration method by finding $y_{m,n+1}$ in terms of values at times n and $n-1$.

Define the dimensionless parameter

$$s = c\frac{\Delta t}{\Delta x}$$

and explore the behavior of your iterative method for the following initial conditions: (a) a sine wave at the zeroth mode; (b) a sine wave at a higher harmonic; and (c) a triangular shape (for a "plucked" string) — and for different values of s. For what values of s is the iteration stable?

Add quadratic coupling terms as suggested in the Fermi-Pasta-Ulam paper [2] (the **FPU 55** link on the syllabus). It's up to you to figure out how to relate their Eq. 1 to your difference equation. Do you observe recurrence after mixing? What does it sound like? Try to correlate the sonic result as closely as possible with the verbal description in [2]. Gould and Tobochnik [3] remark, "[Fermi, Pasta, and Ulam's] surprising discovery might have been the first time a qualitatively new result, instead of a more precise number, was found from a computer simulation."

Extra Credit (Any order)

A. Experiment with different forms of nonlinear coupling and study how that affects the recurrence phenomenon.

Many of the following extensions are from [4]. There is enough here altogether for a classic Master's Thesis – in fact, Ruiz's Master's Thesis [5]. If you have time for extra work on this problem, pick any that interest you.

B. Real strings are subject to a damping force which accounts for dissipation of energy in the form of heat. To a first approximation, the damping force is proportional to the velocity of the string, so the wave equation is modified to become

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - b\frac{\partial y}{\partial t}, \qquad b > 0$$

where b is a constant representing the amount of damping. Incorporate this term into your iteration formula, and explore its effect. How are different harmonics affected? Do higher frequencies die out faster or slower than lower harmonics?

C. Repeat the previous part, but with the damping term of the form

$$+r\frac{\partial^3 y}{\partial t^3}, \qquad r > 0$$

which Hiller and Ruiz claim is the simplest one that can account for loss of energy through radiation. How are the different harmonics now affected?

D. (Stiff string) When stiffness of the string is taken into account, the wave equation becomes

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - S\frac{\partial^4 y}{\partial x^4}, \qquad S > 0$$

where S is a constant depending on properties of the string like Young's modulus of elasticity, cross-sectional area, etc. Try incorporating this into your difference equation. What effect does stiffness have on the fundamental frequency and higher partials of the vibration? (When harmonics are no longer integer multiples of a fundamental frequency, they're called *partials*.)

E. Put together the three additional terms from Parts B-D.

F. Play with the effect of different initial conditions. Can you make the string sound plucked? Bowed?

Bibliography

[1] * G. D. Smith. *Numerical Solution of Partial Differential Equations.* Clarendon Press, Oxford, U.K., third edition, 1985.

[2] E. Fermi, J. Pasta, and S. Ulam. Studies of non linear problems. In *Collected Papers of E. Fermi*, volume II. University of Chicago Press, 1965.

[3] * H. Gould and J. Tobochnik. *An Introduction to Computer Simulation Methods.* Addison Wesly, Menlo Park, Ca., second edition, 1996.

[4] L. Hiller and P. Ruiz. Synthesizing musical sounds by solving the wave equation for vibrating objects. *Journal of the Audio Engineering Society*, 19(6 and 7):462–470 (Part I), 542–551 (Part II), June and August 1971.

[5] P. Ruiz. A technique for simulating the vibrations of strings with a digital computer. Master's thesis, University of Illinois, Champaign-Urbana, Ill., 1969.

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