Appl. Math. Lett. Vol. 4, No. 6, pp. 23–26, 1991
Printed in Great Britain.
0893-9659/91 \$3.00 + 0.00

Filter Automata Admitting Oscillating Carrier Waves

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(Received March 1991)

Abstract. Group-valued cellular automata with soliton behavior are considered and a Fast Rule theorem is proved. This new class of automata generalizes those recently introduced by Fokas, Papadopoulou and Saridakis [1].

INTRODUCTION

One-dimensional cellular automata with solition behavior were introduced by Park, Steiglitz and Thurston [2]. As dynamical systems these automata are, in principle, systems of infinite range, therefore it was a great step forward when the analysis of these so called *Filter Automata* was considerably simplified by discovery of the *fast rule theorem* (FRT) by Papatheodorou, Ablowitz and Saridakis [3]. The FRT not only simplifies the determination of the dynamics, but also presents the mathematical structure which is responsible for the soliton behavior of these systems.

Later these automata were intensively investigated [4, 5] and the soliton structure of these systems was based on a firm theoretical foundation. In a recent paper [1], it was shown that even a generalization to states taking values in finite groups is possible. This is important insofar as it allows the extension to multidimensional systems (of finite width, however).

In this paper we push the applicability of the important fast rule theorem even further. Not only that group-valued states are admitted, but also the carrier wave, that is the underlying evolution of non-boxed states, is now allowed to be of a more complex nature. Furthermore, a wider variety of changes for the so-called boxed states is possible.

The consequences of this extension for the solion behavior are considerable. Not only that more internal degrees of freedom are possible, an effect which will be important for applications, but also the behavior of trivial basic strings (i.e., strings with length 1) is changed drastically. Now, we can have nontrivial basic string which are annihilated, and trivial basic strings may have an oscillatory behavior.

THE LAW OF EVOLUTION

We consider a group G (not necessary abelian). The law of composition in G is denoted by $g \otimes h$, the unit element is e and $\bar{g}$ is the inverse of g. The abbreviation $\otimes_{i=n}^{m} g_i$ stands for $g_n \otimes g_{n+1} \otimes \cdots \otimes g_m$.

We consider one-dimensional lattice vectors $\vec{a} : \mathbf{Z} \to G$ and write these as $\vec{a} = (\dots, a_{-n}, a_{-n+1}, \dots, a_0, \dots, a_n, \dots)$. The manifold under consideration is the set of those lattice vectors being e identically on the left end, i.e., for each such element $\vec{a}$ there is some $m \in \mathbf{Z}$ such that $a_k = e$ for all $k < m$. We are interested in discrete dynamical laws for time dependent lattice vectors $\vec{a}(t)$, where the time t is assumed to run through $\mathbf{Z}$.

We fix some integer $r > 0$, furthermore, group homomorphisms R, ρ and $\tau : G \to G$ and some map $J : G \to G$. The map J cannot be a homomorphism, since we assume $J(e) \neq e$. The maps under consideration are required to fulfill the following compatibility conditions:

$$\overline{R}\tau = \bar{\rho}, \tag{1}$$

$$J\overline{R}J(e) = e, \tag{2}$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX where $\bar{\rho}$ and $\overline{R}$ are the maps $g \to \overline{\rho(g)}$ and $g \to \overline{R(g)}$. The *state* $a_i(t+1)$ is called *exceptional* if the following conditions are fulfilled:

$$\tau(a_i(t)) = \tau(a_{i+1}(t)) = \cdots = \tau(a_{i+r}(t)) = e, \tag{3}$$

$$a_{i-r}(t+1) = a_{i-r+1}(t+1) = \cdots = a_{i-1}(t+1) = e. \tag{4}$$

Observe that, by use of (1), (3) implies that $\rho(a_i(t)) = \cdots = \rho(a_{i+r}(t)) = e$. On the manifold we consider the discrete flow

$$a_i(t+1) = \begin{Bmatrix} e & \text{if } a_i(t+1) \text{ is exceptional} \\ J(s_i(t)) \otimes \tau(a_{i+r}(t)) & \text{otherwise} \end{Bmatrix}, \tag{5}$$

where

$$s_i(t) = \otimes_{j=-r}^{-1} \left\{ \rho(a_{i+j+r}(t)) \otimes \overline{R}(a_{i+j}(t+1)) \right\}. \tag{6}$$

Observe that $s_i(t) = e$ if $a_i(t+1)$ is exceptional. We split this flow up into its leading term and then into a linear and a nonlinear contribution. The *leading term* is the one in (5) being farthest to the right side. By this we obtain a separation of influence. This leads to the introduction of the following quantities γ (*change-function*) and N (*nonlinearity*)

$$a_i(t+1) = \gamma_{i+r}(t) \otimes \tau(a_{i+r}(t)), \tag{7}$$

$$a_i(t+1) = N_{i+r}(t) \otimes J(s_i(t)) \otimes \tau(a_{i+r}(t)). \tag{8}$$

Comparing these two equations we obtain

$$\gamma_{i+r}(t) = N_{i+r}(t) \otimes J(s_i(t)). \tag{9}$$

We say that the state $a_k(t)$ changes by $\gamma_k(t)$. The quantity $\Lambda := J(e)$ is said to be the *standard change*. Later on, we will show that the only values γ attains are e and Λ. The *unchanging* dynamics given by $a_i(t+1) = \tau(a_{i+r}(t))$ we call the *carrier wave* of the system (5). For the analysis of the dynamics we adapt the following notions:
The state $a_k(t)$—at time t—is said to be

(i) a *unit state* if $\tau(a_k(t)) = e$
(ii) a *linear state* if $N_k(t) = e$
(iii) a *non-changing state* if $\gamma_k(t) = e$

ANALYSIS OF STATES

Inserting (7) into (6) we obtain the identity

$$s_i(t) = \otimes_{j=-r}^{-1} \left\{ \rho(a_{i+j+r}(t)) \otimes \overline{R}(\tau(a_{i+j+r}(t))) \otimes \overline{R}(\gamma_{i+r+j}(t)) \right\}. \tag{10}$$

Using (1) this simplifies considerably

$$s_i(t) = \otimes_{j=-r}^{-1} \left\{ \overline{R}(\gamma_{i+r+j}(t)) \right\}. \tag{11}$$

Observation.

(i) If $a_i(t+1)$ is exceptional then, by the evolution law, we have $a_i(t+1) = e$ and, by definition, $\tau(a_{i+r}(t)) = e$. As consequence of (7), $a_{i+r}(t)$ is non-changing. And, using (9) together with $s_i(t) = e$, we find that $N_{i+r}(t) = \overline{J(e)} = \overline{\Lambda}$. Hence, N only attains the values e and $\overline{\Lambda}$.

(ii) If $a_{k-r}(t+1)$ is not exceptional, then by the evolution law (5), we have $N_k(t) = e$. Hence $a_k(t)$ must be linear. Conversely, if $a_k(t)$ is linear, then $N_k(t) \neq \overline{\Lambda}$, and, by the above, $a_{k-r}(t+1)$ cannot be exceptional. So exceptional states correspond uniquely to nonlinear states.

(iii) Let $a_k(t)$ be nonlinear. Since $a_{k-r}(t+1)$ is then exceptional, we obtain from (3) and (4) that

$$\tau(a_k(t)) = e \quad \text{and} \quad \tau(a_{k-r}(t)) = \tau(a_{k-r+1}(t)) = \cdots = \tau(a_{k-1}(t)) = e, \tag{12}$$

$$\gamma_{k-r}(t) = \gamma_{k-r+1}(t) = \cdots = \gamma_{k-1}(t) = e, \tag{13}$$

where the last identities were obtained by inserting (7) and (12) into (4).

We obtain from these observations the following rules:

RULE 1: A state is nonlinear if and only if it is a unit state and is preceded by r non-changing unit states.

RULE 2: If $a_k(t)$ is preceded by r non-changing states, then $a_k(t)$ is changing if and only if it is a linear state. In that case it changes by the standard change Λ.

RULE 3: If for $a_k(t)$ there is among the preceding r states exactly one changing state and if that changes by the standard change, then $a_k(t)$ is non-changing.

PROOF:
1: Assume that $a_k(t)$ is nonlinear, then the implication follows directly from (12) and (13). The converse follows from the fact that if the unit state $a_k(t)$ is preceded by r non-changing unit states then (12) and (13) follow. So, from (3) and (4) we see that $a_{k-r}(t+1)$ is exceptional which is, by (ii), equivalent to $a_k(t)$ being nonlinear.
2: From (11) follows $s_{k-r}(t) = e$, and by (9), $N_k(t) = \gamma_k(t) \otimes \overline{\Lambda}$. By (i) and (ii) nonlinear states are non-changing. So, $a_k(t)$ is changing if and only if it is linear, and from $N_k(t) = e$ it follows $e = \gamma_k(t) \otimes \overline{\Lambda}$. This implies that it changes by the standard change.
3: If $a_k(t)$ is not linear then by (i) and (ii) it is not changing anyway. So, we assume that $a_k(t)$ is linear. Let $a_{k-n}(t)$ the one changing among the r preceding states, then by (11), $s_{k-r}(t) = \overline{R}(\gamma_{k-n}(t)) = \overline{R}(\Lambda) = \overline{R}(J(e))$. With (9) this gives $\gamma_k(t) = J(\overline{R}(J(e)))$. By assumption (2) this is equal to e and $a_k(t)$ is proved to be non-changing. ∎

These rules imply the following generalization of the *Fast Rule Theorem* [3] for obtaining from the configuration $\vec{a}(t)$ the states at time $(t+1)$:

(α) From non-changing states $a_k(t)$ we obtain the $a_{k-r}(t+1) = \tau(a_k(t))$ from the carrier wave.

(β) The connected sequence of unit states to the left end of $\vec{a}(t)$ consists of non-changing states, and the first non-unit state from the left is the first changing state.

(γ) All changing states are changing by the standard change Λ. From one changing state, the next changing state, if there is any, is found by the following:

- Go $(r+1)$ states to the right. If either this state or at least one among the last r states is a non-unit state, then this state is the next changing state. Otherwise the first non-unit state to the right is the next changing state.

PROOF: By definition the unit states to the left are nonlinear, hence non-changing. The first non-unit state to the left must be linear, since nonlinear states are unit states. So, Rule 2 gives that this state changes by the standard change.

Now, by iterating Rule 3 we find that the next r states do not change. Hence, the next candidate for a changing state is the one $(r+1)$ places to the right. This state, by Rule 2, either is changing by Λ and linear or it is nonlinear. If it is nonlinear then by Rule 1 it must be a unit state, and the states in between must be unit states as well. Since nonlinear states are not changing, Rule 2 must in that case provide the next linear state as changing state (which then is changing by Λ). By Rule 1 this state must be the first non-unit state. Repeating the last sequence of arguments we find that the *Fast Rule Theorem* is proved. ∎

Example 1. Fix some non-unit element c in G, define $J(g) := g \otimes c$, put the maps $R = \tau = \rho$ equal to the identity map. Then the compatibility conditions are fulfilled and the situation of [1] is recovered.

Example 2. Consider $G = \mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$ the congruence classes modulo q. Let $c \neq 0$ and $\hat{R}$, $\hat{\rho}$ be elements of $\mathbb{Z}_q$ such that $(\hat{R}+1)c = (\hat{R}+1)\hat{\rho} = 0$. Define $J(z) := z + c$, $R(z) := -\hat{R}z$ and $\tau(z) = \rho(z) := \hat{\rho}z$ then again the compatibility conditions are fulfilled. Automata of this type indeed admit a new soliton behavior. For example: Take $r = 2$, $q = 4$, $\hat{R} = 1$, $\hat{\rho} = 2$, and $c = 2$, then one easily sees that the trivial basic string $0, \cdots, 0, 2, 0, 2, 0, \cdots, 0$ is annihilated after two time-steps, whereas the 1 in $0, \cdots, 0, 1, 0, \cdots, 0$ oscillates between 1 and 3. A complete description of solitons will be given in a forthcoming paper.

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