Computer Science 341
Discrete Mathematics

Midterm Exam

Due at the beginning of class on Wednesday, November 9, 2005

There are 5 problems on this exam. All carry the same weight.
Collaboration Policy: Do not collaborate on the midterm exam.

Problem 1

Express each of the following summations as $\Theta(f(n))$ where $f(n)$ is an appropriate function of n in closed form.

a) $\sum_{k=1}^{n}(1.01)^k$.

b) $\sum_{k=1}^{n}\frac{1}{\sqrt{k}}$.

c) $\sum_{k=1}^{n}\frac{1}{k\log k}$.

d) $\sum_{k=1}^{n}\left(1+\frac{1}{k}\right)^k$.

Problem 2

The running time $T(n)$ of a newly developed algorithm satisfies the following recurrence relation:

$$T(n) = 4T(n/2) - 4T(n/4) + n \qquad T(1) = 1 \qquad T(2) = 4.$$

Find a closed form expression for the running time of this algorithm for instances of size $n = 2^k$, where k is an integer.

Problem 3

Pick a random natural number r between 0 and 1000000: $0 \leq r < 1000000$. What is the probability that the sum of its digits is divisible by 10?

Problem 4

Consider a sequence of n independent tosses of a fair coin. A *run* is defined to be a maximal sequence of contiguous tosses that are either all heads or all tails. e.g. the sequence HHTHHTTTHH has 5 runs of length 2,1,2,3 and 2 respectively.

a) Compute the probability that there are exactly k runs.

b) Compute the probability that there are exactly k runs and every run is of length at most 2.

Problem 5

Two coins are placed in a bag. One of them is a fair coin, i.e. it comes up heads with probability 1/2 and tails with probability 1/2. The other is a special coin with tails on both sides. One of the coins is picked from the bag at random and this coin is tossed n times.

Let A_i be the event that the coin comes up tails on the i^{th} toss.

a) Calculate $\Pr[A_i]$.

b) Calculate $\Pr[A_2|A_1]$.

c) Calculate $\Pr[A_k|A_1 \cap A_2 \cap \ldots \cap A_{k-1}]$.