Computer Science 341
Discrete Mathematics

Final Exam
Due 24 hours from time obtained

Guidelines: No collaboration is permitted on the final. You may refer to your own notes, textbooks (including those other than the two course texts), and all materials posted on the course web site.

Any fact/lemma/theorem in the course materials and course texts may be used in your solutions without proof. If you use material from a textbook other than the course texts, you should include an appropriate citation with your solution; additionally the use of any fact/lemma/theorem from these additional sources must be accompanied with a proof.

There are 5 problems on this exam. Since students will be taking the exam at different times, please do not discuss the questions on the final after you are done.

The solution for each problem must begin on a new page. Be concise and clear.

All the best!

Problem 1 [15 points]

A Huffman code for 3 symbols consists of the codewords 0, 10 and 11. Find the number of $\{0, 1\}$ strings of length n that are valid encodings with these codewords. In other words, find the number of distinct strings of length n that can be obtained by concatenating (copies of) the strings 0, 10 and 11. For example, 0101110 is a valid string (0 10 11 10) but 1100101 is not.

Problem 2 [20 points]

Consider a cycle of length n with vertices $v_1, \ldots, v_n$. v_i is adjacent to v_{i+1} for $1 \leq i \leq n-1$ and v_n is adjacent to v_1. We would like to assign one of k colors to each vertex in the cycle such that adjacent vertices get different colors. Find the number of ways of doing this (as a function of n and k).

Problem 3 [20 points]

A game is played on an $n \times n$ board, as follows: Here (i, j) denotes the square in row i and column j. The goal is to move a special piece (Tiger) from the bottom left corner square $(1, 1)$ to the upper right corner square (n, n). In each move, Tiger can move one square up or one square to the right, i.e. from (i, j), he can move to $(i + 1, j)$ or $(i, j + 1)$. To complicate matters, some squares on the board are marked as forbidden and Tiger is not allowed to land on them. At the beginning of the game, each square on the board (other than $(1, 1)$ and (n, n)) is marked as forbidden at random with probability $1/2$. These choices are made independently for every square. Let p_n be the probability that Tiger can move from $(1, 1)$ to (n, n) avoiding any forbidden squares. Find $\lim_{n\to\infty} p_n$.

Hint: Think of the paths from $(1, 1)$ to (n, n).

Problem 4 [25 points]

A random tree on n vertices is formed in the following way: The vertices are added to the tree in the sequence $v_1, \ldots, v_n$. The first vertex v_1 is considered the root. The ith vertex v_i is added to the tree by connecting it to one of the previous $i-1$ vertices chosen uniformly and at random from amongst $v_1, \ldots, v_{i-1}$. Let P_i be the length of the path (i.e. number of edges on the path) from v_i to the root.

(a) [5 points] Let $R_n = \mathbf{E}[P_n]$. Write a recurrence relation for R_n.

(b) [7 points] Use the recurrence for R_n to compute $\mathbf{E}[P_n]$.
Write down an exact expression (not necessarily in closed form) and also find a closed form function $f(n)$ such that $\mathbf{E}[P_n] = \Theta(f(n))$.

(c) [5 points] Let $S_n = \mathbf{E}[(P_n)^2]$. Write a recurrence relation for S_n.

(d) [8 points] Use the recurrence for S_n to compute $E[(P_n)^2]$.
Write down an exact expression (not necessarily in closed form) and also find a closed form function $g(n)$ such that $\mathbf{E}[(P_n)^2] = \Theta(g(n))$

Problem 5 [20 points]

Consider a sequence of graphs constructed as follows. G_1 consists of a single vertex and G_2 consists of two vertices connected by an edge. Graph G_{i+1} is constructed from G_i by adding some vertices and edges as follows. Suppose G_i has vertex set $V = \{v_1, \ldots, v_n\}$. Then G_{i+1} is constructed by adding vertices $U \cup \{x\}$ where $U = \{u_1, \ldots, u_n\}$. In addition to the edges already present in G_i, the following edges are added to G_{i+1}: For every i, u_i is connected to x and u_i is connected to every neighbor of v_i. Note that G_3 obtained from this process is the 5-cycle. The goal of this problem is to analyze the number of colors in a proper coloring of G_k. Recall that a proper coloring of a graph is an assignment of colors to vertices so that adjacent vertices get different colors.

(a) [5 points] Prove that G_k has a proper coloring with at most k colors.

(b) [15 points] Prove that any proper coloring of G_k must use at least k colors.
Hint: If G_{k+1} can be colored with k colors, show that G_k can be colored with $k-1$ colors.