Computer Science 341
Discrete Mathematics

Final Exam

Guidelines: No collaboration is permitted on the final. You may refer to your own notes, textbooks (including those other than the two course texts), and all materials posted on the course web site. The use of any other electronic materials and searching for information online is not allowed.

Any fact/lemma/theorem in the course materials and course texts may be used in your solutions without proof. If you use material from a textbook other than the course texts, you should include an appropriate citation with your solution; additionally the use of any fact/lemma/theorem from these additional sources must be accompanied with a proof.

There are 5 problems and each problem is worth 20 points. You have 24 hours to complete the exam. Since students will be taking the exam at different times, please do not discuss the questions on the final after you are done.

The solution for each problem must begin on a new page. Be concise and clear.

All the best!

Problem 1 [20 points]

A. [14 points] Solve the following recurrences:

1. $a_{n+2} = 5a_{n+1} + 24a_n$, where $a_0 = 3$, $a_1 = 13$;
2. $a_{n+2} = 22a_{n+1} - 121a_n$, where $a_0 = 3$, $a_1 = 22$;

B. [6 points] Prove that

$$\sum_{i=0}^{n} |\sin(i)| = \Theta(n)$$

Remark: The argument of $\sin(\cdot)$ is in radians. Recall that π radians = 180 degrees.

Problem 2 [20 points]

The goal of this problem is to prove the following formula

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \cdot 2^{n-2k} = n+1$$

A. [8 points] Let M be the set of strings of length n over $\{0,1\}$. For every i: $1 \le i \le n-1$, let

$$A_i = \{(x_1, x_2, \ldots, x_n) \in M : x_i = 0, x_{i+1} = 1\}.$$

Prove, that

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n-1} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = \binom{n-k}{k} \cdot 2^{n-2k}.$$

B. [8 points] Prove that $|M \setminus \bigcup_{1 \le i \le n-1} A_i| = n+1$.

Hint: Find a simple description of strings not contained in any set A_i.

C. [4 points] Conclude that

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \cdot 2^{n-2k} = n+1$$

Problem 3 [20 points]

A random graph on n vertices can be built as follows: for each pair of verices (i, j), toss a fair coin; if the outcome is heads, add an edge between i and j; otherwise, don't. Let T_n be the number of triangles (cycles of length 3) in the graph. Note that T_n is a random variable – a function of the coin tosses used to construct the graph.

A. [5 points] Find the expectation of T_n.

B. [8 points] Find the variance of T_n.

C. [7 points] Let p_n be the probability that every vertex of the graph is covered by a triangle, i.e. for every vertex i, there is a triangle in the graph containing i. Prove that p_n tends to 1 as n tends to infinity.

Hint: Bound the probability that a single vertex is not covered by a triangle.

Problem 4 [20 points]

Given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we produce graph $G = (V, E)$, where $V = \{\langle u, v \rangle : u \in V_1, v \in V_2\}$, $\langle u_1, v_1 \rangle$ and $\langle u_2, v_2 \rangle$ are connected by an edge *iff* either $\langle u_1, u_2 \rangle \in E_1$ or $\langle v_1, v_2 \rangle \in E_2$ (including the case when both conditions are true). Show that

A.[10 points] $\alpha(G) \ge \alpha(G_1) \cdot \alpha(G_2)$

B.[10 points] $\alpha(G) \le \alpha(G_1) \cdot \alpha(G_2)$

Remark: $\alpha(G)$ is the size of the maximum independent set of G – the size of the largest subset I of vertices such that no two vertices in I have an edge between them.

Problem 5 [20 points]

A model recently proposed in social networks to model the effect of distance on social relationships is the following: Consider a graph where the vertices are integer points (x, y) such that $x \in \{0, \ldots, n-1\}$ and $y \in \{0, \ldots, n-1\}$. Place an edge between two points (x_1, y_1) and (x_2, y_2) with probability

$$p(x_1, y_1, x_2, y_2) = \left(\frac{1}{\max(|x_1 - x_2|, |y_1 - y_2|)} \right)^{\alpha},$$

where $\alpha > 0$ is a constant. Let X_n be the degree of the vertex $(0, 0)$. Give an asymptotic expression for the expectation $\mathbf{E}[X_n]$, i.e. find a function $f(n)$ in closed form such that $\mathbf{E}[X_n] = \Theta(f(n))$.

Hint: Find the number of points (x, y) such that $\max(x, y) = i$.