COS 341: Discrete Mathematics

Midterm Exam | Fall 2006

Print your name____

General directions: This exam is due on **Monday, November 13 at 4:30pm.** Late exams will *not* be accepted. Exams must be submitted in hard copy to my office, room 407 of the Computer Science Building. (If I am not around, you can slide your exam under my door.) **Please attach this cover page** as well as the following page with the grading table (preferably printed back-to-back) when you turn in your exam, and be sure to **sign the pledge** below.

During the exam, **you may not collaborate or discuss the exam or any course material** with fellow classmates, friends, TA's or anyone else. Of course, you can ask me to clarify specific questions if they are unclear or ambiguous.

During the exam, you may use only the following materials:

1. your own lecture notes;
2. the Lehman & Leighton notes;
3. other miscellaneous materials that may have been distributed, such as homeworks, precept problem sets, or email help messages sent by the course staff.

In addition, in case you need to review relevant material covered in MAT 103/104, the prerequisites for this course, you may use "Karl's Calculus Tutor" at www.karlscalculus.org, which includes such topics as factoring polynomials (in Prependix C), limits (Chapter 2), and the method of partial fractions (Section 11.7). Alternatively, you can use another standard single-variable calculus textbook; if you do, please indicate on your exam which book you are using. (Let me know if you are aware of a better on-line calculus resource.)

Other than what is listed above, **you may not use any other materials**, including books, articles, materials from previous years or other courses, or any materials taken from the web.

On all of the questions below, unless indicated otherwise, **be sure to show your work and justify all your answers**. As on the homeworks, this means to give an explanation of why your answers are correct and legitimate. In some cases, a proof is called for, a more stringent requirement demanding a complete, rigorous, formal, logical argument. In general, the only facts, lemmas, theorems, etc. that can be used without proof are the ones provided in lecture or the Lehman & Leighton notes. The grading criteria will be similar to what is described for homeworks on the course "Assignments" web page. Approximate point values are given in brackets.

Good luck!

Honor pledge: "I pledge my honor that I have not violated the Honor Code during this examination." **Please write the pledge in full and sign:**

Please turn in this page, but do not write on it – for grader use only.

1		______	/ 10
2	a	______	/ 10
	b	______	/ 2
3	a	______	/ 10
	b	______	/ 3
4	a	______	/ 5
	b	______	/ 5
	c	______	/ 5
5	a	______	/ 7
	b	______	/ 6
TOTAL:		______	/ 63

1. [10] The United States Senate has 100 seats, two for each of the 50 states. Senators are elected to six-year terms, with roughly a third of the senate seats coming up for election every two years. Thus, each senate seat is assigned to one of the following three election cycles:

- A: $\ldots, 1994, 2000, 2006, 2012, \ldots$
- B: $\ldots, 1996, 2002, 2008, 2014, \ldots$
- C: $\ldots, 1998, 2004, 2010, 2016, \ldots$

where each list indicates the years in which that seat comes up for election. For instance, New Jersey's two senate seats happen to be on election cycles A and B, which means one of the two senate seats (the one on cycle A) is up for election this year, and the other seat (on cycle B) will next be up for election in 2008. Moreover, the same state will never have both its senate seats up for election in the same year; for instance, New Jersey cannot have both its seats assigned to cycle B.

To summarize, in assigning senate seats to election cycles, the following constraints must be satisfied:

- each of the 50 states is assigned two seats;
- every seat must be assigned to one of the six-year-term election cycles above;
- no state can have both seats assigned to the same election cycle;
- each election cycle must have either 33 or 34 seats assigned to it.

Note that the two senate seats of any one state are indistinguishable. For instance, New Jersey's two seats, for our purposes, are identical, so that assigning one seat to cycle A and the other to cycle B, is exactly the same as assigning one to cycle B and the other to A.

Given these constraints, exactly how many ways are there to assign senate seats to the three election cycles? Your final answer can involve factorials and binomial coefficients.

2. Let H_n denote the n-th harmonic number.

a. [10] Let

$$g(n) = \frac{1}{n+1}\sum_{i=1}^{n} H_i.$$

Give an exact closed form expression for $g(n)$. Your answer may itself involve harmonic numbers. Prove the correctness of your answer.

b. [2] For which of the following choices of $f(n)$, if any, is it the case that $g(n) = \theta(f(n))$?

$$n,\ \ \log\log n,\ \ n\log n,\ \ (\log n)^2,\ \ n^2,\ \ \log n,\ \ n/\log n,\ \ 2^n$$

Briefly justify your answer.

3. Consider the following code for sending messages over the five-letter alphabet a, b, c, d, e:

$$\begin{aligned} a &= 00 \\ b &= 01 \\ c &= 10 \\ d &= 110 \\ e &= 111. \end{aligned}$$

For instance, the message "*ebbc*" would be encoded by the 9-bit binary string "111010110". (It can be verified that an encoded binary message produced in this way can always be uniquely decoded back into the original message over the five-letter alphabet. You do *not* need to prove this fact, but can take it as given.)

As a further example, using 5-bit binary strings, there are exactly 12 complete messages that can be encoded, namely:

$$\begin{array}{ll} ad = 00110 & ae = 00111 \\ bd = 01110 & be = 01111 \\ cd = 10110 & ce = 10111 \\ da = 11000 & ea = 11100 \\ db = 11001 & eb = 11101 \\ dc = 11010 & ec = 11110 \end{array}$$

a. [10] In general, exactly how many complete messages can be encoded using n-bit binary strings, for $n \geq 1$? Give a closed form expression.

b. [3] As n becomes very large, approximately what fraction of all possible n-bit binary strings correspond to actual complete messages under this code? Briefly justify your answer.

4.

a. [5] Let $B(x)$ be the generating function for some sequence $\langle b_0, b_1, b_2, \ldots \rangle$. Let

$$c_n = b_0 + b_1 + \cdots + b_n = \sum_{i=0}^{n} b_i,$$

and let $C(x)$ be the generating function for this sequence $\langle c_0, c_1, c_2, \ldots \rangle$. Find an exact closed form expression for $C(x)$ in terms of x and $B(x)$.

b. [5] Let $G(x)$ be the generating function for some sequence $\langle g_0, g_1, g_2, \ldots \rangle$. Let f_n be defined by the recurrence:

$$\begin{aligned} f_0 &= g_0 \\ f_n &= g_n + \sum_{i=0}^{n-1} f_i \quad (n \geq 1), \end{aligned}$$

and let $F(x)$ be the generating function for this sequence $\langle f_0, f_1, f_2, \ldots \rangle$. Find an exact closed form expression for $F(x)$ in terms of x and $G(x)$.

c. [5] Suppose in the last part that $g_n = (-1)^n$. Use your answer to the last part to derive an exact closed form expression for f_n in terms of n.

5. Suppose the Munsters are giving out k kinds of candy on Halloween, and that each child is allowed to choose at most r pieces.

a. [7] By considering the number of possible combinations of candy that can be selected, give a combinatorial proof that

$$\sum_{i=0}^{r} \binom{i+k-1}{k-1} = \binom{r+k}{k}.$$

b. [6] Reprove this identity directly using induction and algebraic manipulation.