COS 341: Discrete Mathematics

Final Exam — Fall 2006

Print your name____

General directions: This exam is due on **Monday, January 22 at 4:30pm.** Late exams will *not* be accepted. Exams must be submitted in hard copy to my office, room 407 of the Computer Science Building. (If I am not around, you can slide your exam under my door.) **Please attach this cover page** as well as the following page with the grading table (preferably printed back-to-back) when you turn in your exam, and be sure to **sign the pledge** below.

During the exam, **you may not collaborate or discuss the exam or any course material** with fellow classmates, friends, TA's or anyone else. Of course, you can ask me to clarify specific questions if they are unclear or ambiguous.

During the exam, you may use only the following materials:

1. your own lecture notes;
2. the Lehman & Leighton notes;
3. other miscellaneous materials that may have been distributed, such as homeworks, precept problem sets, miscellaneous readings or email help messages sent by the course staff.

In addition, in case you need to review relevant material covered in MAT 103/104, the prerequisites for this course, you may use "Karl's Calculus Tutor" at www.karlscalculus.org, which includes such topics as factoring polynomials (in Prependix C), limits (Chapter 2), and the method of partial fractions (Section 11.7). Alternatively, you can use another standard single-variable calculus textbook; if you do, please indicate on your exam which book you are using. (Let me know if you are aware of a better on-line calculus resource.)

Other than what is listed above, **you may not use any other materials**, including books, articles, materials from previous years or other courses, or any materials taken from the web.

On all of the questions below, unless indicated otherwise, **be sure to show your work and justify all your answers**. As on the homeworks, this means to give an explanation of why your answers are correct and legitimate. In some cases, a proof is called for, a more stringent requirement demanding a complete, rigorous, formal, logical argument. In general, the only facts, lemmas, theorems, etc. that can be used without proof are the ones provided in lecture, precept or the Lehman & Leighton notes. The grading criteria will be similar to what is described for homeworks on the course "Assignments" web page. Approximate point values are given in brackets.

Good luck!

Honor pledge: "I pledge my honor that I have not violated the Honor Code during this examination." **Please write the pledge in full and sign:**

Please turn in this page, but do not write on it – for grader use only.

1	a	______	/ 5
	b	______	/ 7
2	a	______	/ 3
	b	______	/ 4
	c	______	/ 2
	d	______	/ 3
	e	______	/ 5
3		______	/ 10
4	a	______	/ 9
	b	______	/ 9
	c	______	/ 9
5		______	/ 11
TOTAL:		______	/ 77

1. Sam has two coins, F and B. Coin F is a fair coin, i.e., it comes up heads and tails with equal probability. Coin B is a biased coin that comes up heads with probability 3/4, and tails with probability 1/4. Sam flips the coins according to the following process. For the very first flip, Sam flips coin F. Then, for each subsequent flip, Sam continues to flip the coin he is currently using with probability 4/5, or switches to the other coin with probability 1/5. Thus, for example, since coin F was used for the first flip, he will flip the same coin F again for the second flip with probability 4/5, but will switch to flipping coin B for the second flip with probability 1/5. (His decision of which coin to flip only depends on which coin was used last; it is not influenced, for instance, by the outcome of previous flips or which coins were used prior to the previous round.)

Let X_t be a random variable indicating which coin (F or B) was used for the t-th flip, and let Y_t be a random variable indicating the outcome of the t-th flip (heads or tails).

a. [5] Given the description above, indicate whether each of the statements below are true or false. (Your justifications for your answers to this part can be brief and somewhat informal, but should still be convincing.)

(i) Y_3 and Y_5 are independent.

(ii) Y_3 and Y_5 are conditionally independent given Y_4.

(iii) Y_3 and Y_5 are conditionally independent given X_4.

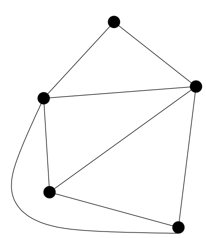
(iv) Y_4 and Y_5 are conditionally independent given X_3.

(v) X_3 and Y_5 are conditionally independent given X_5.

b. [7] Suppose the first flip comes up tails, and the next two flips are heads. What is the probability that coin F was used for the third flip?

2. Consider a simple graph G that is generated randomly on a set of n vertices as follows: for each distinct pair of vertices u and v, an edge $\{u, v\}$ is included in the graph with probability 1/2 (independent of all other edges).

In a graph, a *triangle* is a cycle of length three. Let X be the random variable denoting the number of distinct triangles in the random graph G. (Note that two triangles are considered distinct even if they have one or two vertices in common.) For instance, this graph has five triangles:



a. [3] Compute the expected value of X. Give a general expression in terms of n, and also evaluate this expression numerically when $n = 1000$.

b. [4] Compute the variance of X. Give a general expression in terms of n, and also evaluate the standard deviation of X numerically when $n = 1000$.

c. [2] Use Markov's inequality to compute an upper bound on the probability that X exceeds its expectation by 10% (i.e., that X is at least $1.1\mathrm{E}[X]$). Give a general expression in terms of n, and also evaluate your bound numerically when $n = 1000$.

d. [3] Use Chebyshev's inequality to compute an upper bound on the probability that X exceeds its expectation by 10%. Give a general expression in terms of n, and also evaluate your bound numerically when $n = 1000$.

e. [5] Use Chernoff bounds to compute an upper bound on the probability that X exceeds its expectation by 10%. Give a general expression in terms of n, and also evaluate your bound numerically when $n = 1000$. For this part, you can use any of the Chernoff bounds discussed in class, precept or the readings, provided that the assumptions of the bound you are using are appropriately satisfied. (For these purposes, we will count both Hoeffding's inequality and McDiarmid's inequality as Chernoff bounds.)

3. [10] Let n be a positive integer. Let A be a sequence of n decimal digits, where each digit is selected uniformly and independently at random from $\{0, \ldots, 9\}$. Let B be the reverse of A. Finally, regarding A and B as decimal integers, let S be their sum. (For instance, if $n = 4$ and $A = 0743$ then $B = 3470$ and $S = 4213$. Or if $n = 4$ and $A = 1790$ then $B = 0971$ and $S = 2761$.)

As a function of n, compute the exact probability that S is divisible by 11.

4. Below are a list of pairs of statements, each pair consisting of one true and one false statement. For each pair, prove that one of the two statements is true, and prove that the other statement is false (be sure to clearly indicate which one is true and which is false).

For instance, a pair might be something like the following:

z. (i) For every integer n, if n is even, then $n + 1$ is even.

(ii) For every integer n, if n is even, then $n + 1$ is odd.

Then your job would be to prove that (ii) is true and to give a second proof that (i) is false.

a. [9]

(i) For any two real-valued random variables X and Y,

$$\mathrm{E}[\min\{X, Y\}] \le \min\{\mathrm{E}[X], \mathrm{E}[Y]\}.$$

(ii) For any two real-valued random variables X and Y,

$$\mathrm{E}[\min\{X, Y\}] \ge \min\{\mathrm{E}[X], \mathrm{E}[Y]\}.$$

b. [9]

(i) For any two positive integers a and b, if a and b are relatively prime then $\gcd(a+b, a-b)$ divides 2.

(ii) For any two positive integers a and b, if a and b are relatively prime then $\gcd(a+b, a-b)$ is divisible by 2.

c. [9] The *complement* of a graph $G = (V, E)$ is the graph $G' = (V, E')$ where $\{u, v\} \in E'$ if and only if $\{u, v\} \notin E$. That is, G' has the same vertex set as G, and includes exactly those edges that do *not* occur in G. (As usual, we only consider simple graphs.)

(i) For any graph G, if G is connected, then its complement is not connected.

(ii) For any graph G, if G is not connected, then its complement is connected.

5. [11] Teams A and B are competing against one another for the world championship of the new sport bayesball. According to the arcane rules of such a championship tournament, the two teams continue to play game after game against one another until one of the teams has won W games more than the other team, where W is a small positive integer. (Ties are not possible in bayesball.)

Suppose that teams A and B are very evenly matched so that each team is equally likely to win the first game. Further, since psychological momentum is so important in this sport, assume that the winner of each game has a $2/3$ chance of winning the next game. For instance, if team A wins game 4, then team A will go on to also win game 5 with probability $2/3$, while team B will have only a $1/3$ chance of winning game 5.

What is the expected number of games of bayesball that will be played until the tournament finally ends?

(Hint: Focus on the difference between the number of games won by team A and the number of games won by team B. Initially, this difference is zero, and with each game, it increases or decreases by one. The tournament ends when this difference reaches W or $-W$.)