COS 341: Discrete Mathematics

Precept #8 — Fall 2006
For the week of: December 4

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1. Suppose we attempt to solve the Towers of Hanoi puzzle randomly. How long will it take? To keep it simple, suppose there are only two disks which begin on post #1, and that the goal is to move both disks, according to the usual rules, to either post #2 or post #3. Further, assume that when faced with a choice of legal moves, one is selected uniformly at random. How long, in expectation, will it take to solve the puzzle?

2. Let $X_1, \ldots, X_n$ be mutually independent, but not necessarily identical random variables with respective ranges $R_1, \ldots, R_n$. Let $f(x_1, \ldots, x_n)$ be a function on $R_1 \times \cdots \times R_n$ such that perturbing x_i changes f by at most some constant c_i. That is, for all $x_1, \ldots, x_n$ and for x_i' in the appropriate ranges,

$$|f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)| \leq c_i.$$

McDiarmid's inequality states that under these conditions, for $\epsilon > 0$,

$$\Pr[f(X_1, \ldots, X_n) \geq \mathrm{E}[f(X_1, \ldots, X_n)] + \epsilon] \leq \exp\left(-\frac{2\epsilon^2}{c_1^2 + \cdots + c_n^2}\right).$$

Show how Hoeffding's inequality follows as a special case of McDiarmid's inequality, and also show how Hoeffding can be generalized when the random variables involved are independent and bounded, but not necessarily Bernoulli or identically distributed.

3. All quantities below are integers.

a. Prove that for all n and all $d > 0$, there exists a unique pair q and r such that $n = dq + r$ and $0 \leq r < d$.

b. In class and the notes, we showed that $\gcd(a, b)$ is equal to the smallest linear combination of a and b. Generalize this result to a sequence of numbers $a_1, \cdots, a_n$.