COS 341: Discrete Mathematics

Precept #5 — Fall 2006

For the week of: October 23

---

1. Consider the summation:

$$\sum_{k=0}^{n} k \binom{n}{k}.$$

Give a combinatorial interpretation of this summation and use it to obtain a closed form expression for it.

2. In earlier precepts, we considered the *Catalan numbers* C_n which satisfy the recurrence

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

with initial condition $C_0 = 1$. Previously, we saw that C_n describes, for instance, the number of ways of parenthesizing the product of $n+1$ numbers.

a. Find the generating function for this sequence in closed form.

b. Use this generating function to show that $C_n = \binom{2n}{n}/(n+1)$. For this purpose, you might find it helpful to use the *extended binomial theorem*, an extention of the standard binomial theorem for raising a binomial to an arbitrary real power. This theorem states that, for any real number u,

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k,$$

where we define

$$\binom{u}{k} = \frac{u(u-1)(u-2)\cdots(u-k+1)}{k!}$$

for u any real number, and k any natural number. (Technically, this theorem may not hold when $|x| \geq 1$, but this does not matter for our purposes.)

3. An ordered partition of a natural number n is an ordered sequence of positive natural numbers $(a_1, a_2, \ldots, a_k)$ which add up to exactly n. For instance, there are four ordered partitions of the number 3:

$$1+1+1$$
$$1+2$$
$$2+1$$
$$3$$

Use generating functions to determine the number of ordered partitions of any natural number n.