COS 341: Discrete Mathematics

Homework #5 — Fall 2006

Combinatorial proof and generating functions — Due: Friday, October 27

Special late policy: Because this homework is due right before break, there will be a special late policy in which the Saturday, Sunday and Monday following the deadline count together as a single late "day." To be clear, this table shows how many "days" late your assignment will be counted if turned in on the following dates:

Calendar date	**Number of late days charged**
Friday, October 27	0
Monday, October 30	1
Tuesday, October 31	2
Wednesday, November 1	3
Thursday, November 2	*Not accepted on or after this date*

For this homework only, if you are out of the Princeton area over break, you may mail or email your assignment to Siddhartha. If mailed, your homework is considered submitted on the post mark date, and should be sent to this address: Siddhartha Brahma, Princeton University, Department of Computer Science, 35 Olden Street, Princeton, NJ 08540. (It would be wise to send him email at the same time you mail your assignment so that he can look out for it; also, save a photocopy of your work.) Note that mailing your assignment may delay when it is graded and returned to you.

See instructions on the "assignments" web-page on how and when to turn in homework, and be sure to read the collaboration and late policy for this course. Approximate point values are given in parentheses. Be sure to show your work and justify your answers.

1. (15) The goal of this problem is to prove the following identity:

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \cdot 2^{n-2k} = n+1. \tag{1}$$

a. Recall that $\{0,1\}^n$ is the set of all bit strings of length n. For every i: $1 \le i \le n-1$, let

$$A_i = \{(x_1, x_2, \ldots, x_n) \in \{0,1\}^n : x_i = 0, x_{i+1} = 1\}.$$

Thus, A_i is the set of length-n bit strings with 0 in position i and 1 in position $i+1$. Prove, that

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n-1} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = \binom{n-k}{k} \cdot 2^{n-2k}.$$

(Here, it is understood that the sum is taken over all sequences satisfying the inequality under the $\sum$.)

b. Prove that $|\{0,1\}^n - \bigcup_{1 \le i \le n-1} A_i| = n+1$. (Here, $S - T$ denotes the set of elements that *are* in set S, but are *not* in set T.)
Hint: Find a simple description of strings not contained in any set A_i.

c. Prove the identity given in Eq. (1) above.

2. (12) Miss McGillicuddy never goes outside without a collection of pets. In particular:

- She brings 3, 4, or 5 dogs.
- She brings a positive number of songbirds, which always come in pairs.
- She may or may not bring her alligator, Freddy.

Let s_n denote the number of different collections of n pets that can accompany her. For example, $s_6 = 2$ since there are 2 possible collections of 6 pets:

- 3 dogs, 2 songbirds, 1 alligator
- 4 dogs, 2 songbirds, 0 alligators.

a. Give a closed-form generating function for the sequence $\langle s_0, s_1, s_2, s_3, \ldots \rangle$.

b. From this generating function, find a closed-form expression for s_n. (Your answer may involve several cases.)

3. (12) In this problem, we use generating functions to determine the number of ways to make change for n cents using pennies, nickels, dimes and quarters. (For instance, there are two ways to make change for 7 cents — using 7 pennies, or using 2 pennies and 1 nickel.)

a. Find, in closed-form, a generating function for which the coefficient of x^n is equal to the number of ways to make change for n cents using only pennies. Repeat this step for each of the three other coin types.

b. Find, in closed form, a generating function for which the coefficient of x^n is equal to the number of ways to make change for n cents using pennies, nickels, dimes and quarters.

c. Explain how to use this function from the last part to determine the number of ways to change 99 cents; you do *not* have to provide the answer or actually carry out the process.

4. (15) In this problem, we will use generating functions to solve the recurrence:

$$\begin{aligned} t_0 &= 0 \\ t_1 &= 1 \\ t_n &= 5t_{n-1} - 6t_{n-2} + n^2 \quad (\text{for } n \geq 2) \end{aligned}$$

a. Find a closed-form generating function $F(x)$ for the sequence $\langle t_0, t_1, t_2, \ldots \rangle$.

b. Rewrite this generating function as a sum of fractions of the form:

$$\frac{A}{(1-\alpha x)^k}$$

where A and α are real (or potentially complex) constants, and k is a natural number.

c. Find a closed form for the coefficients of each of the fractions $A/(1-\alpha x)^k$, and use the addition rule to obtain a closed-form expression for t_n.