1. (15) The goal of this problem is to prove the following identity:

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \cdot 2^{n-2k} = n + 1. \tag{1}
\]

a. Recall that \( \{0,1\}^n \) is the set of all bit strings of length \( n \). For every \( i: 1 \leq i \leq n - 1 \), let

\[
A_i = \{(x_1, x_2, \ldots, x_n) \in \{0,1\}^n : x_i = 0, x_{i+1} = 1\}.
\]

Thus, \( A_i \) is the set of length-\( n \) bit strings with 0 in position \( i \) and 1 in position \( i + 1 \).

Prove, that

\[
\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n-1} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = \binom{n-k}{k} \cdot 2^{n-2k}.
\]

(Here, it is understood that the sum is taken over all sequences satisfying the inequality under the \( \sum \).)

b. Prove that \( |\{0,1\}^n - \bigcup_{1 \leq i \leq n-1} A_i| = n + 1 \). (Here, \( S-T \) denotes the set of elements that are in set \( S \), but are not in set \( T \).)

Hint: Find a simple description of strings not contained in any set \( A_i \).

c. Prove the identity given in Eq. (1) above.
2. (12) Miss McGillicuddy never goes outside without a collection of pets. In particular:
   • She brings 3, 4, or 5 dogs.
   • She brings a positive number of songbirds, which always come in pairs.
   • She may or may not bring her alligator, Freddy.

Let \( s_n \) denote the number of different collections of \( n \) pets that can accompany her. For example, \( s_6 = 2 \) since there are 2 possible collections of 6 pets:
   • 3 dogs, 2 songbirds, 1 alligator
   • 4 dogs, 2 songbirds, 0 alligators.

a. Give a closed-form generating function for the sequence \( \langle s_0, s_1, s_2, s_3, \ldots \rangle \).

b. From this generating function, find a closed-form expression for \( s_n \). (Your answer may involve several cases.)

3. (12) In this problem, we use generating functions to determine the number of ways to make change for \( n \) cents using pennies, nickels, dimes and quarters. (For instance, there are two ways to make change for 7 cents — using 7 pennies, or using 2 pennies and 1 nickel.)

   a. Find, in closed-form, a generating function for which the coefficient of \( x^n \) is equal to the number of ways to make change for \( n \) cents using only pennies. Repeat this step for each of the three other coin types.

   b. Find, in closed form, a generating function for which the coefficient of \( x^n \) is equal to the number of ways to make change for \( n \) cents using pennies, nickels, dimes and quarters.

   c. Explain how to use this function from the last part to determine the number of ways to change 99 cents; you do not have to provide the answer or actually carry out the process.

4. (15) In this problem, we will use generating functions to solve the recurrence:

\[
\begin{align*}
t_0 &= 0 \\
t_1 &= 1 \\
t_n &= 5t_{n-1} - 6t_{n-2} + n^2 \quad (\text{for } n \geq 2)
\end{align*}
\]

a. Find a closed-form generating function \( F(x) \) for the sequence \( \langle t_0, t_1, t_2, \ldots \rangle \).

b. Rewrite this generating function as a sum of fractions of the form:

\[
\frac{A}{(1 - \alpha x)^k}
\]

where \( A \) and \( \alpha \) are real (or potentially complex) constants, and \( k \) is a natural number.

c. Find a closed form for the coefficients of each of the fractions \( A/(1 - \alpha x)^k \), and use the addition rule to obtain a closed-form expression for \( t_n \).