COS 341: Discrete Mathematics

Precept #4 — Fall 2006

For the week of: October 16

1.

a. Count the number of sequences of the form $(A_1, \ldots, A_k)$ where $A_1, \ldots, A_k$ are each subsets of $\{1, \ldots, n\}$, and where

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k.$$

b. Count the number of sequences of the form $(A_1, \ldots, A_k)$ where $A_1, \ldots, A_k$ are each subsets of $\{1, \ldots, n\}$, and where

$$A_1 \cup A_2 \cup \cdots \cup A_k = \{1, \ldots, n\}.$$

2. A *subsequence* of a sequence $(a_1, a_2, \ldots, a_n)$ is another sequence of the form $(a_{i_1}, a_{i_2}, \ldots, a_{i_k})$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

Show that in any sequence of $n^2 + 1$ distinct integers, there must be either an increasing subsequence of length $n + 1$, or a decreasing subsequence of length $n + 1$.

3. As on problem 2 from last week's precept, let us consider paths in the plane that only go up or to the right by single unit-length steps. We will say that such a path is *tame* if it never goes above the diagonal line $y = x$ (although such a path *is* allowed to touch that diagonal line). A path that is not tame is said to be *wild*. By an unresticted path, we mean a path that can be of either kind, wild or tame. Our goal is to determine C_n, the number of tame paths beginning at $(0, 0)$ and ending at (n, n).

a. How many unrestricted paths are there which begin at (a, b) and end at (n, m)?

b. Describe a bijection from the set of all wild paths which begin at $(0, 0)$ and end at (n, n) to the set of all unrestricted paths which begin at $(-1, 1)$ and end at (n, n).

c. Compute C_n.