COS 341: Discrete Mathematics

Homework #3 — Fall 2006
Linear recurrences and beginning counting — Due: Friday, October 13

See instructions on the "assignments" web-page on how and when to turn in homework, and be sure to read the collaboration and late policy for this course. Approximate point values are given in parentheses. Be sure to show your work and justify your answers.

1. (10) A sequence of drinks are prepared as follows: Drink 0 consists of plain water; drink 1 consists of wine straight from the bottle; and for $n = 2, 3, \ldots$, drink n is prepared by mixing equal parts of the last two drinks (i.e., drink $n - 1$ and drink $n - 2$). In general, what fraction of drink n will be wine, and what part will be water? What do these fractions converge to as n becomes large?

2. (10) Let a, b and n be natural numbers, and let

$$f(n) = \frac{(a + \sqrt{b})^n + (a - \sqrt{b})^n}{2}.$$

a. Find a homogeneous linear recurrence that is satisfied by $f(n)$.

b. Use your answer to part (a) to prove that $f(n)$, as defined above, is a natural number.

3. (10) The 16 contestants in an ice-cream eating contest are lined up and numbered in order $0, \ldots, 15$. Remarkably, not including the two contestants sitting at either end of the line, it is discovered when the contest is over that each contestant n has eaten one ounce more than the average of his or her two immediate neighbors, contestants $n - 1$ and $n + 1$. Suppose contestant 0 ate 14 ounces of ice cream, and contestant 15 ate 2 ounces of ice cream. How much did each of the other contestants eat? Who won the contest?

4. (15)

a. Find the number of ways of tiling a $2 \times n$ rectangle with 1×2 tiles, so that there is no overlap and the rectangle is entirely covered.

b. Find the number of ways of tiling a $3 \times n$ rectangle with 1×2 tiles, so that there is no overlap and the rectangle is entirely covered.

5. (10) There are 20 books arranged in a row on a shelf. Describe a bijection between ways of choosing 6 of these books so that no two adjacent books are selected, and 15-bit sequences with exactly 6 ones.