COS 341: Discrete Mathematics

Homework #2 — Fall 2006
Sums, asymptotics, recurrences — Due: Friday, October 6

See instructions on the "assignments" web-page on how and when to turn in homework, and be sure to read the collaboration and late policy for this course. Approximate point values are given in parentheses.

1. (10) To buy a house, George borrows $P_0 = \$100{,}000$ from a bank. The *principal* on the loan is the amount of money still owed, so initially the principal is P_0. The interest on the loan is 7%. This means that at the end of each year, the principal increases by $r = 0.07$ times whatever the principal was at the beginning of the year. Also, at the very end of each year, George pays the bank a fixed amount of $\$x$.

Give your answers below in terms of variables (P_0, r, etc.) rather than actual numbers (100000, 0.07, etc.), except as indicated in the last part.

a. Give a recurrence describing the principal P_t at the end of the t-th year.

b. Solve the recurrence in the last part to give a closed form expression for P_t, and prove the correctness of your answer.

c. How should the annual payment of $\$x$ be chosen so that George will have exactly paid off the loan at the end of $N = 30$ years? Give your answer in terms of variables, but also evaluate it numerically for the specific values given above.

2. (10) Use integration to find upper and lower bounds that differ by at most 0.1 for the following sum. (You may need to add the first few terms explicitly and then use integrals to bound the sum of the remaining terms.)

$$\sum_{i=1}^{\infty} \frac{1}{(2i+1)^2}.$$

3. (10) Consider a variant of the Towers of Hanoi problem in which the disks cannot be moved directly between post #1 and post #3, but must instead first be moved via post #2. In other words, all moves must involve post #2, either moving a disk to that middle post, or removing it from the middle post.

a. Let $T(n)$ denote the optimal number of moves to transfer a stack of n disks from post #1 to post #3. Give a recurrence for $T(n)$, and prove its correctness. (This means giving an algorithm that takes only $T(n)$ moves, and also arguing that every algorithm for this problem must take at least this many moves.)

b. Solve this recurrence, and prove the correctness of your answer.

4. (10) For the following, you can assume that all functions take only positive values.

a. Prove that if $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$ then

$$f_1(n) + f_2(n) = O(\max\{g_1(n), g_2(n)\}).$$

b. Suppose that $f(n) = o(g(n))$, meaning that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

Prove that $f(n) = O(g(n))$, and that $f(n) \neq \Omega(g(n))$.

5. (10) Determine which of these choices

$$\theta(1),\ \ \theta(n),\ \ \theta(n^2 \log n),\ \ \theta(n^2),\ \ \theta(2^n),\ \ \theta(4^n),\ \ \theta(2^{n \ln n}),\ \ \text{none of these}$$

describes each function's asymptotic behavior. Proofs are not required, but briefly explain your answers.

a. $(n^2 + 2n - 3)/(n^2 - 7)$

b. $\sum_{i=0}^{n} 2^{2i+1}$

c. $\ln((n^2)!)$

d. $\sum_{k=1}^{n} k \left(1 - \frac{1}{2^k}\right)$

e. $1 + n(\sin(n\pi/2))^2$