COS 341: Discrete Mathematics

Homework #10 — Fall 2006
Graph Theory — Due: Friday, January 12

See instructions on the "assignments" web-page on how and when to turn in homework, and be sure to read the collaboration and late policy for this course. Approximate point values are given in brackets. *Be sure to show your work and justify your answers.*

1.

a. [8] Let $G = (V, E)$ be a graph, and let $\deg(v)$ denote the degree of a vertex v. Find an exact closed form expression for

$$\sum_{v \in V} \deg(v)$$

in terms of $|V|$ and $|E|$. Give a careful, formal proof by induction of your answer.

b. [3] Show that at any party with an odd number of people, at least one person shakes an even number of hands.

2. A *cycle* is a path that begins and ends at the same vertex. That is, a cycle is a sequence of vertices $x_0, x_1, \ldots, x_n$ such that $\{x_i, x_{i+1}\}$ is an edge of the graph, and such that $x_0 = x_n$, and $x_1, \ldots, x_n$ are all distinct. The length of such a cycle is n. For instance, a triangle is a cycle of length 3.

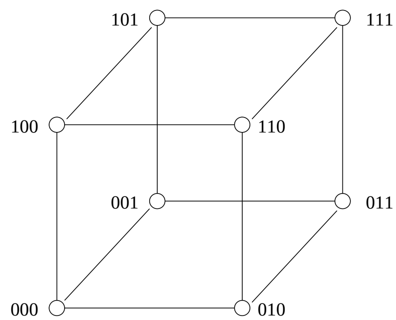
A *matching* in a graph is a subset M of the edges such that no two edges in M are incident on a common vertex.

a. [8] Prove that a graph is 2-colorable if and only if it contains no odd-length cycles.

b. [7] Let M_1 and M_2 be two matchings of a graph $G = (V, E)$. Consider the new graph $G' = (V, M_1 \cup M_2)$ (i.e., on the same vertex set, whose edges consist of all the edges that appear in either M_1 or M_2). Prove that G' is 2-colorable.

3. [8] Suppose that in some graph, the degree of every vertex is at least k, where $k \geq 2$. Prove that the graph must have a cycle of length at least $k + 1$. (As usual, we assume that the graph is simple with no self-loops or multiple edges.)

4. A network of 2^n computers are connected to each other as follows: each computer is assigned a unique n-bit sequence and two computers are connected by a direct link if their corresponding bit sequences differ in exactly one position. For instance, if $n = 3$, then there would be eight computers identified as 000, 001, 010, 011, 100, 101, 110, 111, and connected in a network as shown below:



If a machine fails, all links incident on it are unavailable for communication. We would like the network to be fault-tolerant in the sense that it should remain connected even if there are a small number of machine failures. In this problem, we will show that the minimum number of machine failures required to disconnect this network is exactly n.

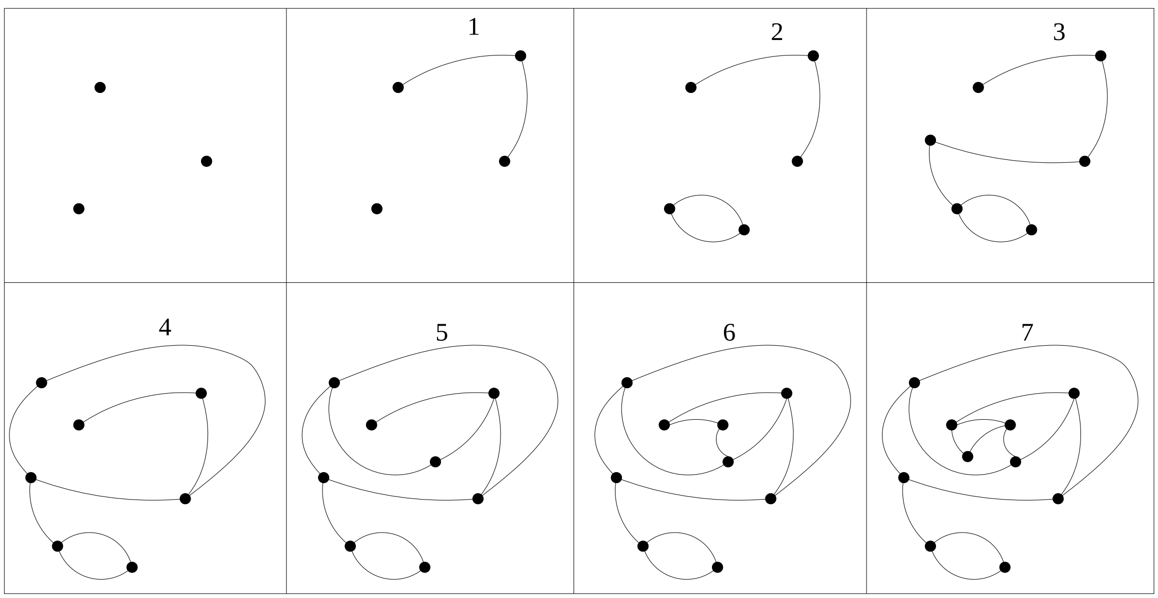
a. [3] Show that there exist n machines which, when removed, cause the network to no longer be connected.

b. [6] Let $\mathbf{0}_n$ denote the node $000\cdots0$ in the network whose assigned bit sequence is all 0's, and similarly define $\mathbf{1}_n$ to denote the node $111\cdots1$. Show that there exist n disjoint paths from $\mathbf{0}_n$ to $\mathbf{1}_n$ (i.e., n paths from $\mathbf{0}_n$ to $\mathbf{1}_n$ so that no node of the network appears on more than one of the n paths).

c. [6] Show that between *any* two nodes of the network, there exist n disjoint paths. (You may (or may not) wish to use the last part for this.) Explain why this shows that at least n machines must be removed to disconnect the network.

5. [8] *Sprouts* is a two-player game played using paper and pencil. Initially, n vertices are drawn on the paper with no edges connecting them, where n is typically between 2 and 10. On each turn, the player whose turn it is: (1) draws a new vertex, (2) selects two (not necessarily distinct) existing vertices, and (3) draws two new edges connecting the new vertex to the two selected vertices. Multiple edges connecting the same two vertices are allowed. However, each play is subject to the following rules: First, no edge may ever cross, overlap or pass through itself or any other edge or vertex (although of course it will connect to the vertices to which it is incident). Similarly, no vertex may be placed so that it coincides with another edge or vertex. Thus, the graph will at all times be planar. Second, no vertex may ever have degree greater than three. A player loses the game when he or she is stuck with no moves possible from the current configuration.

An example sequence of plays is shown below with $n = 3$. In this case, the game ends after seven moves so that the second player, who is unable to move in the final configuration, has lost.



One might guess at first that a game of sprouts can go on forever. Prove, on the contrary, that every game of sprouts must end (so that no moves are possible from the final configuration) by the time at most $3n - 1$ moves have been played.