COS 341: Discrete Mathematics

Precept #1 Fall 2006
For the week of: September 25

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1. Consider the game *chomp* as described on the homework assignment. Suppose the game is played on a square $n \times n$ grid, with $n \geq 2$. Find a winning strategy for Alice. That is, show how Alice can play in such a way that she is guaranteed to win.

2. There are two children sitting on a (very long) bench. The child on the left is a boy, the child on the right is a girl. Every minute, either two children arrive and sit down next to each other on the bench (possibly squeezing between two children who are already sitting), or two children who had been sitting next to each other get up off the bench and leave. Furthermore, the arriving and departing pairs of children are always of the same sex (i.e., either both boys or both girls).

Is it possible, after some amount of time, that there will be only two children remaining on the bench with a girl on the left and a boy on the right?

Hint: Try to find an *invariant*, i.e., a property of the boy-girl ordering of the children that does not change with time.

3. Suppose we have a box that initially contains some finite (but possibly very large) number of billiard balls. Each ball is numbered with a positive integer, called the "rank" of the ball. At each time step, we select a single ball to remove from the box, and replace it with any number of other balls, each assigned whatever rank we wish, with the restriction that all of the new balls have rank strictly smaller than the rank of the ball that was removed. For instance, we might choose to remove a single ball of rank 983 from the box, and replace it with a thousand balls of rank 847, a billion balls of rank 711, and a trillion balls of rank 982. However, when a ball of rank 1 is removed, it is not replaced with anything.

Given a choice of the balls initially in the box, and given a choice of which balls are removed and how they are replaced, is it possible to cause this process to go on forever? Or must it eventually end after a finite (but possibly extremely long) length of time?

4. This problem develops the beginnings of a discrete analog of continuous calculus. For any function f defined on the natural numbers, let us define the function Δf by

$$\Delta f(n) = f(n+1) - f(n).$$

This is the analog of a derivative. A discrete sum is the analog of the integral.

a. Show that

$$\sum_{i=a}^{b} \Delta f(i) = f(b+1) - f(a).$$

This is the analog of the fundamental theorem of calculus.

b. For any positive integer k, let

$$n^{(k)} = n(n-1)(n-2)\cdots(n-k+1).$$

This is the discrete analog of n^k. Show that

$$\Delta n^{(k)} = kn^{(k-1)}.$$

c. Using parts (a) and (b), what is

$$\sum_{i=0}^{n} i^{(k)}$$

in closed form? Does your answer agree with the form we earlier proved for the n-th triangle number (i.e., $\sum_{i=0}^{n} i$)?

d. Show that n^2 can be written in the form

$$n^2 = n^{(2)} + bn^{(1)}$$

for some constant b. Use this, together with part (c), to evaluate

$$\sum_{i=0}^{n} i^2.$$

e. Generalize part (d). In particular, for k a positive integer, show that

$$\sum_{i=0}^{n} i^k = \frac{n^{k+1}}{k+1} + p(n)$$

where $p(n)$ is a polynomial of degree at most k.

f. Try to figure out some of the other discrete analogs of functions and topics from calculus, such as e^x, $\ln x$, $1/x^k$, derivative of a product, integration by parts, etc.