COS 341: Discrete Mathematics

Homework #1 | Fall 2006
Induction | Due: Friday, September 29

See instructions on the "assignments" web-page on how and when to turn in homework, and be sure to read the collaboration and late policy for this course. Approximate point values are given in parentheses.

1. (10) Find a closed form expression for

$$\sum_{i=1}^{n} \frac{1}{i(i+1)}.$$

Be sure to prove your answer.

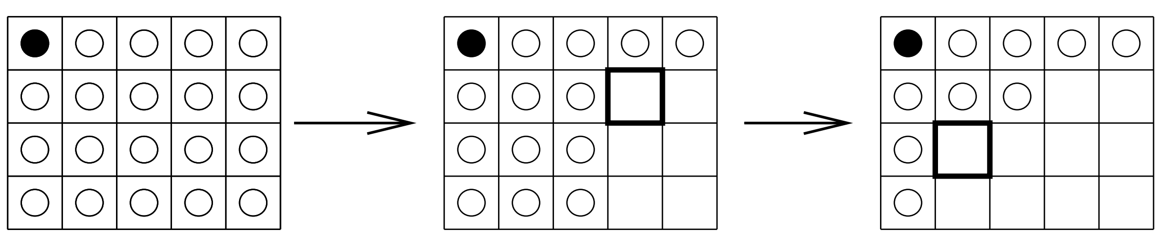
2. (10) Use induction (or strong induction) to prove that it is possible using only 4-cent and 7-cent stamps to form exact postage of n cents for any $n \geq 18$.

3. (10) Consider the sequence of numbers $X_0, X_1, X_2, \ldots$ defined as follows:

$$\begin{aligned} X_0 &= 0 \\ X_1 &= 1 \\ X_n &= 5X_{n-1}^3 + 7X_{n-2} \quad \text{for } n \geq 2. \end{aligned}$$

Prove that if n is a multiple of 3, then X_n is even.

4. (10) The game of *chomp* is played on an $m \times n$ grid between two players, Alice and Bob, with Alice going first. Initially the grid is filled with cookies, as shown in the left figure below, in this case, on a 4×5 grid. The top left cookie is poison, and whoever eats it, loses the game. The players take turns eating cookies, and at least one of the remaining cookies must be eaten on every turn. A player can choose to eat any remaining cookie, but doing so causes all of the remaining cookies below or to the right of the chosen cookie to also be eaten. For instance, in the center figure below, Alice has eaten the cookie in the blackened square, causing the five cookies below or to its right also to be eaten. Similarly, in the right figure, Bob has eaten the cookie in the blackened square of that figure, causing the three remaining cookies below or to its right also to be eaten.



Suppose the game is played on a $2 \times n$ grid (that is, with two rows and n columns), where $n \geq 1$. Find a winning strategy for Alice. In other words, show how Alice can play in such a way that she is guaranteed to win. Be sure to prove that your strategy always causes Alice to win, regardless of how Bob plays.

5. (10) Use induction to prove that the natural numbers are *well ordered*. That is, use induction (or strong induction) to show that for any subset S of $\mathbb{N}$, if S is non-empty, then S has a least element. Or equivalently, you can prove the contrapositive of this statement, namely, that if S does not have a least element, then S is empty.

(We say that x is a *least element* of a set S if $x \in S$ and if $x \leq y$ for every $y \in S$. Not all sets have least elements, for instance, the set of all negative integers, or the set of all (strictly) positive real numbers.)