COS226

Algorithms and Data Structures
Princeton University
Fall 2006

Kevin Wayne

Overview

What is COS 226?

- Intermediate-level survey course.
- Programming and problem solving with applications.
- Algorithm: method for solving a problem.
- Data structure: method to store information.

Topic	Data Structures and Algorithms
data types	stack, queue, list, union-find, priority queue
sorting	quicksort, mergesort, heapsort, radix sorts
searching	hash table, BST, red-black tree, B-tree
graphs	DFS, Prim, Kruskal, Dijkstra, Ford-Fulkerson
strings	KMP, Rabin-Karp, TST, Huffman, LZW
geometry	Graham scan, k-d tree, Voronoi diagram

A misperception: algiros [painful] + arithmos [number].

Impact of Great Algorithms

Internet. Web search, packet routing, distributed file sharing.
Biology. Human genome project, protein folding.
Computers. Circuit layout, file system, compilers.
Computer graphics. Hollywood movies, video games.
Security. Cell phones, e-commerce, voting machines.
Multimedia. CD player, DVD, MP3, JPG, DivX, HDTV.
Transportation. Airline crew scheduling, map routing.
Physics. N-body simulation, particle collision simulation.
…

> For me, great algorithms are the poetry of computation. Just like verse, they can be terse, allusive, dense, and even mysterious. But once unlocked, they cast a brilliant new light on some aspect of computing. - Francis Sullivan

Why Study Algorithms?

Using a computer?

- Want it to go faster? Process more data?
- Want it to do something that would otherwise be impossible?

Algorithms as a field of study.

- Philosophical implications.
- Burgeoning application areas.
- Old enough that basics are known.
- New enough that new discoveries arise.

$$E = mc^2 \qquad F = \frac{G m_1 m_2}{r^2}$$
$$F = ma$$
$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\Psi(r) = E\,\Psi(r)$$

20th century science
(formula based)

bioinformatics
neurosciences
computational physics
…

21st century science
(algorithm based)

The Usual Suspects

Lectures. [Kevin Wayne]

- TTh 11-12:20, Friend 008.

Precepts. [Wolfgang Mulzer, Janet Yoon]

- Th 12:30, Friend 108.
- Th 3:30, Friend 108.
- Discuss programming assignments, exercises, lecture material.
- First precept meets 9/21.

Questionnaire

Please fill out questionnaire so that we can adapt course as needed.

- Who are you?
- Why are you taking COS 226?
- Which precept(s) can you attend?
- What do you hope to get out of it?
- What is your programming experience?

Coursework and Grading

7 programming assignments. 45%

- Due 11:55pm, starting Monday 9/25.
- Available via course website.

Weekly written exercises. 15%

- Due at beginning of Thursday lecture, starting 9/21.
- Available via course website.

Exams.

- Closed book with cheatsheet.
- Midterm. 15%
- Final. 25%

Staff discretion. Adjust borderline cases.

Course Materials

Course web page. `http://www.princeton.edu/~cos226`

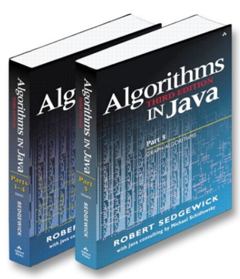
- Syllabus.
- Exercises.
- Lecture slides.
- Programming assignments.

Algorithms in Java, 3rd edition.

- Parts 1-4. [sorting, searching]
- Part 5. [graph algorithms]

Algorithms in C, 2nd edition.

- Strings and geometry handouts.

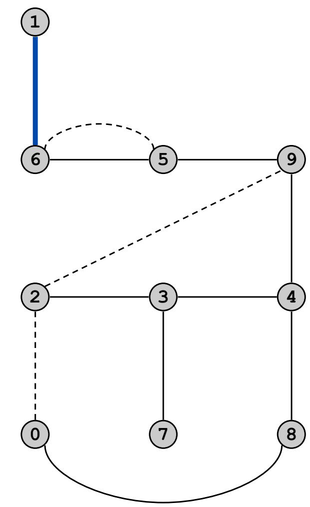


Union Find

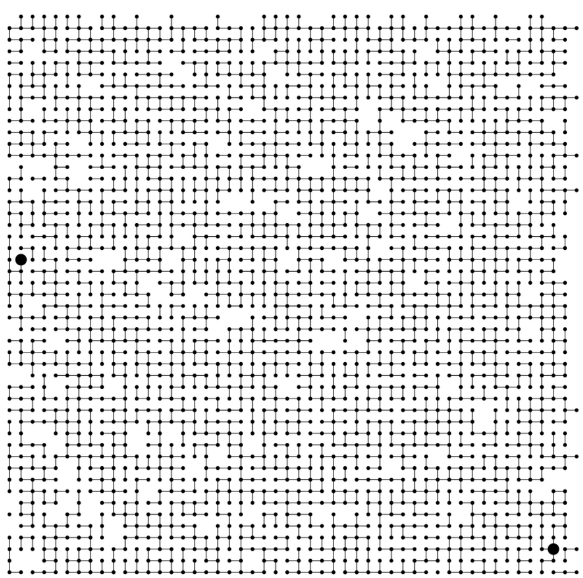
Reference: Chapter 1, Algorithms in Java, 3rd Edition, Robert Sedgewick.

Network Connectivity

in	out	evidence
3 4	3 4	
4 9	4 9	
8 0	8 0	
2 3	2 3	
5 6	5 6	
2 9		(2–3–4–9)
5 9	5 9	
7 3	7 3	
4 8	4 8	
5 6		(5–6)
0 2		(2–3–4–8–0)
6 1	6 1	



Network Connectivity



Union-Find Abstraction

What are critical operations we need to support?

- Objects.

  `0 1 2 3 4 5 6 7 8 9` — grid points

- Disjoint sets of objects.

  `0 1 2-3-9 5-6 7 4-8` — subsets of connected grid points

- Find: are objects 2 and 9 in the same set?

  `0 1 2-3-9 5-6 7 4-8` — are two grid points connected?

- Union: merge sets containing 3 and 8.

  `0 1 2-3-4-8-9 7` — add a connection between two grid points

Union-Find Abstraction

What are critical operations we need to support?

- Objects.
- Disjoint sets of objects.
- Find: are two objects in the same set?
- Union: replace sets containing two items by their union.

Goal. Design efficient data structure for union and find.

- Number of operations M can be huge.
- Number of objects N can be huge.

Objects

Applications involve manipulating objects of all types.

- Variable name aliases.
- Pixels in a digital photo.
- Computers in a network.
- Web pages on the Internet.
- Transistors in a computer chip.
- Metallic sites in a composite system.

When programming, convenient to name them 0 to N-1.

- Details not relevant to union-find.
- Integers allow quick-access to object-related info.

array indices

Quick-Find [eager approach]

Data structure.

- Integer array `id[]` of size `N`.
- Interpretation: `p` and `q` are connected if they have the same id.

i	0	1	2	3	4	5	6	7	8	9
id[i]	0	1	9	9	9	6	6	7	8	9

5 and 6 are connected
2, 3, 4, and 9 are connected

Find. Check if `p` and `q` have the same id.

id[3] = 9; id[6] = 6
3 and 6 not connected

Union. To merge components containing `p` and `q`, change all entries with `id[p]` to `id[q]`.

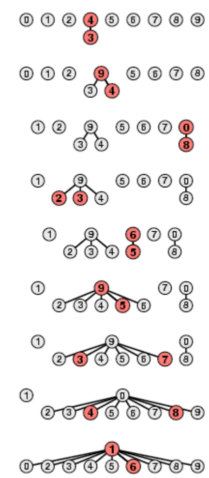
i	0	1	2	3	4	5	6	7	8	9
id[i]	0	1	6	6	6	6	6	7	8	6

union of 3 and 6
2, 3, 4, 5, 6, and 9 are connected

many values can change

Quick-Find: Example

3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	9	9	5	6	7	8	9
8-0	0	1	2	9	9	5	6	7	0	9
2-3	0	1	9	9	9	5	6	7	0	9
5-6	0	1	9	9	9	6	6	7	0	9
5-9	0	1	9	9	9	9	9	7	0	9
7-3	0	1	9	9	9	9	9	9	0	9
4-8	0	1	0	0	0	0	0	0	0	0
6-1	1	1	1	1	1	1	1	1	1	1



Quick-Find: Java Implementation

```
public class QuickFind {
   private int[] id;

   public QuickFind(int N) {
      id = new int[N];
      for (int i = 0; i < N; i++)
         id[i] = i;
   }

   public boolean find(int p, int q) {
      return id[p] == id[q];
   }

   public void unite(int p, int q) {
      int pid = id[p];
      for (int i = 0; i < id.length; i++)
         if (id[i] == pid) id[i] = id[q];
   }
}
```

set id of each object to itself

1 operation

N operations

Problem Size and Computation Time

Rough standard for 2000.
- 10^9 operations per second.
- 10^9 words of main memory.
- Touch all words in approximately 1 second. [unchanged since 1950!]

Ex. Huge problem for quick find.
- 10^{10} edges connecting 10^9 nodes.
- Quick-find might take 10^{20} operations. [~10 ops per query]
- 3,000 years of computer time!

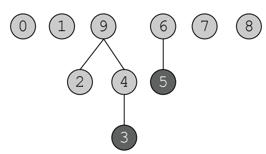
Paradoxically, quadratic algorithms get worse with newer equipment.
- New computer may be 10x as fast.
- But, has 10x as much memory so problem may be 10x bigger.
- With quadratic algorithm, takes 10x as long!

Quick-Union [lazy approach]

Data structure.
- Integer array `id[]` of size `N`.
- Interpretation: `id[i]` is parent of `i`.
- Root of `i` is `id[id[id[...id[i]...]]]`.

keep going until it doesn't change

i	0	1	2	3	4	5	6	7	8	9
id[i]	0	1	9	4	9	6	6	7	8	9



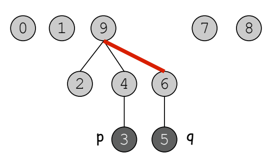
3's root is 9; 5's root is 6
3 and 5 are not connected

Find. Check if p and q have the same root.

Union. Set the id of q's root to the id of p's root.

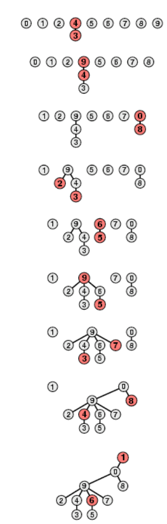
i	0	1	2	3	4	5	6	7	8	9
id[i]	0	1	9	4	9	6	9	7	8	9

only one value changes



Quick-Union: Example

3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	9	5	6	7	8	9
8-0	0	1	2	4	9	5	6	7	0	9
2-3	0	1	9	4	9	5	6	7	0	9
5-6	0	1	9	4	9	6	6	7	0	9
5-9	0	1	9	4	9	6	9	7	0	9
7-3	0	1	9	4	9	6	9	9	0	9
4-8	0	1	9	4	9	6	9	9	0	0
6-1	1	1	9	4	9	6	9	9	0	0



Quick-Union: Java Implementation

```
public class QuickUnion {
   private int[] id;

   public QuickUnion(int N) {
      id = new int[N];
      for (int i = 0; i < N; i++) id[i] = i;
   }

   private int root(int i) {
      while (i != id[i]) i = id[i];
      return i;
   }

   public boolean find(int p, int q) {
      return root(p) == root(q);
   }

   public void unite(int p, int q) {
      int i = root(p);
      int j = root(q);
      id[i] = j;
   }
}
```

time proportional to depth of x

time proportional to depth of p and q

time proportional to depth of p and q

Summary

Quick-find defect.

- Union too expensive.
- Trees are flat, but too expensive to keep them flat.

Quick-union defect.

- Finding the root can be expensive.
- Trees can get tall.

Data Structure	Union	Find
Quick-find	N	1
Quick-union	tree height	N

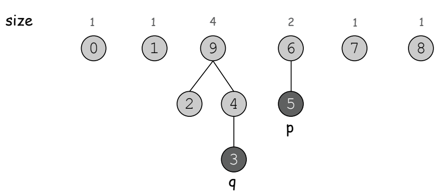
Weighted Quick-Union

Weighted quick-union.

- Modify quick-union to avoid tall trees.
- Keep track of size of each component.
- Balance by linking small tree below large one.

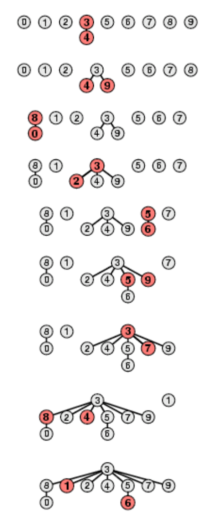
Ex. Union of 5 and 3.

- Quick union: link 9 to 6.
- Weighted quick union: link 6 to 9.



Weighted Quick-Union: Example

3-4	0	1	2	3	3	5	6	7	8	9
4-9	0	1	2	3	3	5	6	7	8	3
8-0	8	1	2	3	3	5	6	7	8	3
2-3	8	1	3	3	3	5	6	7	8	3
5-6	8	1	3	3	3	5	5	7	8	3
5-9	8	1	3	3	3	3	5	7	8	3
7-3	8	1	3	3	3	3	5	3	8	3
4-8	8	1	3	3	3	3	5	3	3	3
6-1	8	3	3	3	3	3	5	3	3	3



Weighted Quick-Union: Java Implementation

Java implementation.

- Almost identical to quick-union.
- Maintain extra array `sz[]` to count number of elements in the tree rooted at i.

Find. Identical to quick-union.

Union. Same as quick-union, but merge smaller tree into larger tree, and update the `sz[]` array.

```
if (sz[i] < sz[j]) { id[i] = j; sz[j] += sz[i]; }
else               { id[j] = i; sz[i] += sz[j]; }
```


Weighted Quick-Union: Analysis

Analysis.

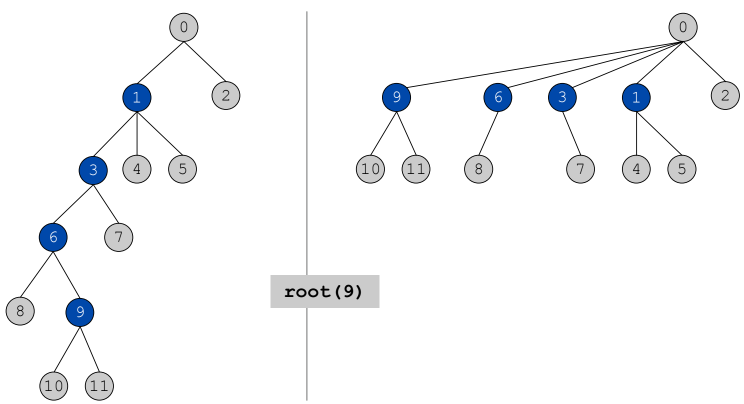
- Find: takes time proportional to depth of `p` and `q`.
- Union: takes constant time, given roots.
- Fact: depth is at most lg N. [needs proof]

Data Structure	Union	Find
Quick-find	N	1
Quick-union	$1^{\dagger}$	N
Weighted QU	lg N	lg N

Stop at guaranteed acceptable performance? No, can improve further.

Path Compression

Path compression. Just after computing the root of `i`, set the `id` of each examined node to `root(i)`.



Weighted Quick-Union with Path Compression

Path compression.

- Standard implementation: add second loop to `root()` to set the id of each examined node to the root.
- Simpler one-pass variant: make every other node in path point to its grandparent.

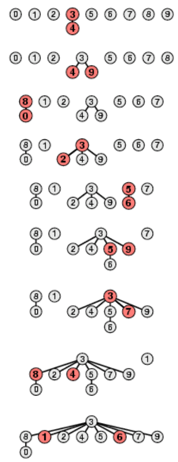
```
public int root(int i) {
   while (i != id[i]) {
      id[i] = id[id[i]];
      i = id[i];
   }
   return i;
}
```

only one extra line of code !

In practice. No reason not to! Keeps tree almost completely flat.

Weighted Quick-Union with Path Compression

3-4	0	1	2	3	3	5	6	7	8	9
4-9	0	1	2	3	3	5	6	7	8	3
8-0	8	1	2	3	3	5	6	7	8	3
2-3	8	1	3	3	3	5	6	7	8	3
5-6	8	1	3	3	3	5	5	7	8	3
5-9	8	1	3	3	3	3	5	7	8	3
7-3	8	1	3	3	3	3	5	3	8	3
4-8	8	1	3	3	3	3	5	3	3	3
6-1	8	3	3	3	3	3	3	3	3	3



Weighted Quick-Union with Path Compression

Theorem. Starting from an empty data structure, any sequence of M union and find operations on N elements takes O(N + M lg* N) time.
- Proof is very difficult.
- But the algorithm is still simple!

Remark. lg* N is a constant in this universe.

N	lg* N
2	1
4	2
16	3
65536	4
2^{65536}	5

Linear algorithm?
- Cost within constant factor of reading in the data.
- Theory: WQUPC is not quite linear.
- Practice: WQUPC is linear.

Context

Ex. Huge practical problem.
- 10^{10} edges connecting 10^9 nodes.
- WQUPC reduces time from 3,000 years to 1 minute.
- Supercomputer won't help much.
- Good algorithm makes solution possible.

Bottom line. WQUPC on Java cell phone beats QF on supercomputer!

Algorithm	Time
Quick-find	M N
Quick-union	M N
Weighted QU	N + M log N
Path compression	N + M log N
Weighted + path	5 (M + N)

M union-find ops on a set of N elements

Applications

Other Applications

Union-find applications.

- Hex.
- Percolation.
- Connectivity.
- Image processing.
- Least common ancestor.
- Equivalence of finite state automata.
- Hinley-Milner polymorphic type inference.
- Kruskal's minimum spanning tree algorithm.
- Compiling equivalence statements in Fortran.

Percolation

Percolation phase-transition.

- Two parallel conducting bars (top and bottom).
- Electricity flows from a site to one of its 4 neighbors if both are occupied by conductors.
- Model: each site is a conductor with probability p.

top

0	0	0	0	0	0	0	0	0	0	0	0
2	3	4	5	6	0	8	9	10	11	12	0
14	15	0	0	0	0	20	21	22	23	24	0
14	14	28	29	30	31	32	33	34	35	36	0
14	39	40	1	42	43	32	45	46	1	1	49
50	1	52	1	54	55	56	57	58	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1

bottom

conductor
insulator

Percolation

Q. What is percolation threshold p* at which charge carriers can percolate from top to bottom?

A. ~ 0.592746 for square lattices.

percolation constant only known via simulation

top

0	0	0	0	0	0	0	0	0	0	0	0
2	3	4	5	6	0	8	9	10	11	12	0
14	15	0	0	0	0	20	21	22	23	24	0
14	14	28	29	30	31	32	33	34	35	36	0
14	39	40	1	42	43	32	45	46	1	1	49
50	1	52	1	54	55	56	57	58	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1

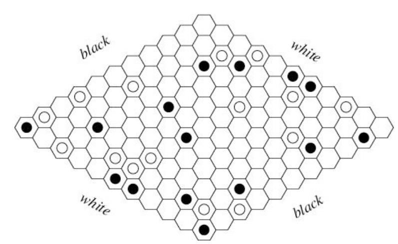
bottom

conductor
insulator

Hex

Hex. [Piet Hein 1942, John Nash 1948, Parker Brothers 1962]

- Two players alternate in picking a cell in a hex grid.
- Black: make a black path from upper left to lower right.
- White: make a white path from lower left to upper right.



Reference: http://mathworld.wolfram.com/GameofHex.html

Goal. Algorithm to detect when a player has won.

Summary

Lessons.

- Start with simple, brute force approach.
  - don't use for large problems
  - can't use for huge problems
- Strive for worst-case performance guarantees. might be nontrivial to analyze
- Identify fundamental abstractions: union-find.
- Apply to many domains.