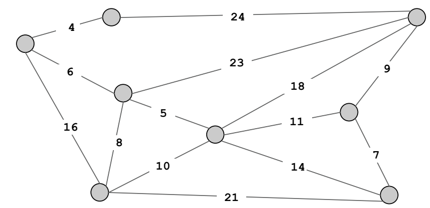
Minimum Spanning Tree

Reference: Chapter 20, Algorithms in Java, 3rd Edition, Robert Sedgewick

Minimum Spanning Tree

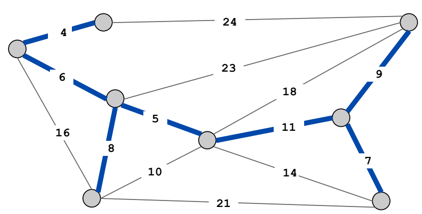
MST. Given connected graph G with positive edge weights, find a min weight set of edges that connects all of the vertices.



G

Minimum Spanning Tree

MST. Given connected graph G with positive edge weights, find a min weight set of edges that connects all of the vertices.



cost(T) = 50

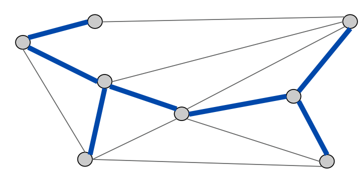
Theorem. [Cayley, 1889] There are V^{V-2} spanning trees on the complete graph on V vertices.

can't solve by brute force

MST Origin

Otakar Boruvka (1926).

- Electrical Power Company of Western Moravia in Brno.
- Most economical construction of electrical power network.
- Concrete engineering problem is now a cornerstone problem in combinatorial optimization.



Otakar Boruvka

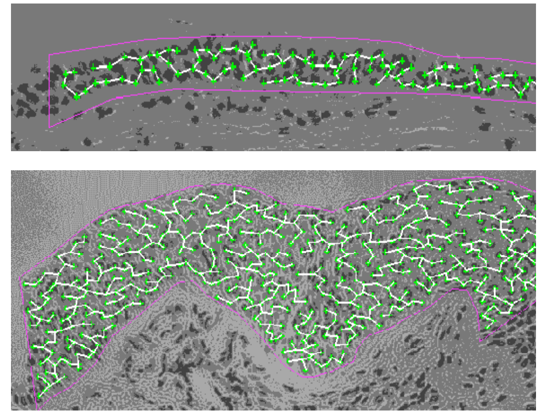
Applications

MST is fundamental problem with diverse applications.

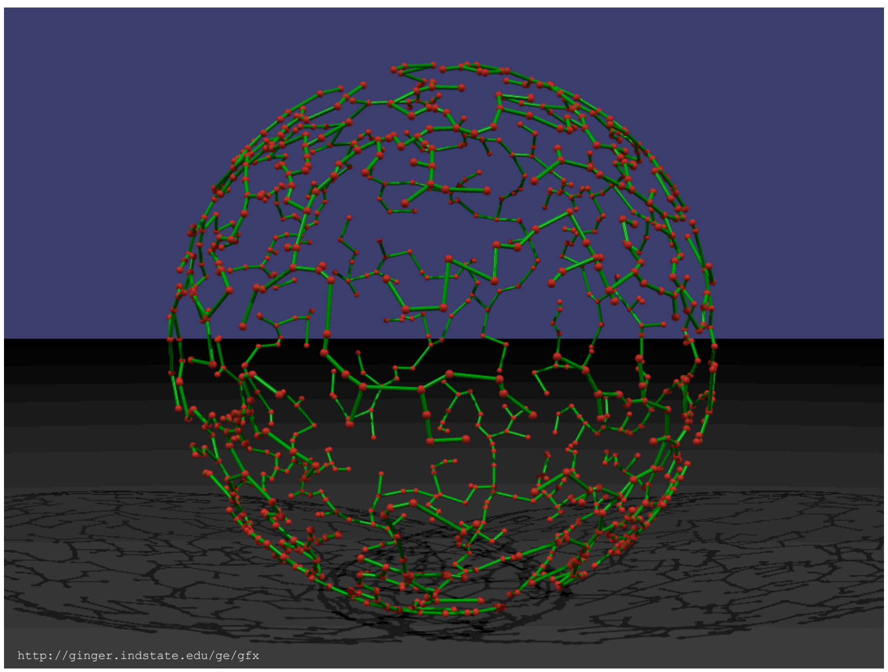
- Network design.
  - telephone, electrical, hydraulic, TV cable, computer, road
- Approximation algorithms for NP-hard problems.
  - traveling salesperson problem, Steiner tree
- Indirect applications.
  - max bottleneck paths
  - LDPC codes for error correction
  - image registration with Renyi entropy
  - learning salient features for real-time face verification
  - reducing data storage in sequencing amino acids in a protein
  - model locality of particle interactions in turbulent fluid flows
  - autoconfig protocol for Ethernet bridging to avoid cycles in a network
- Cluster analysis.

Medical Image Processing

MST describes arrangement of nuclei in the epithelium for cancer research



http://www.bccrc.ca/ci/ta01_archlevel.html



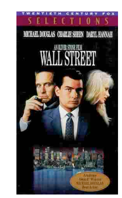
Two Greedy Algorithms

Kruskal's algorithm. Consider edges in ascending order of cost. Add the next edge to T unless doing so would create a cycle.

Prim's algorithm. Start with any vertex s and greedily grow a tree T from s. At each step, add the cheapest edge to T that has exactly one endpoint in T.

Theorem. Both greedy algorithms compute an MST.

Greed is good. Greed is right. Greed works. Greed clarifies, cuts through, and captures the essence of the evolutionary spirit." - Gordon Gecko



Weighted Graphs

Weighted Graph Interface

public class WeightedGraph (graph data type)

	`WeightedGraph(int V)`	create an empty graph with V vertices
`void`	`insert(Edge e)`	insert edge e
`Iterable<Edge>`	`adj(int v)`	return an iterator over edges incident to v
`int`	`V()`	return the number of vertices
`String`	`toString()`	return a string representation

```
for (int v = 0; v < G.V(); v++) {
   for (Edge e : G.adj(v)) {
      int w = e.other(v);
      // edge v-w
   }
}
```

iterate through all edges (once in each direction)

Edge Data Type

```
public class Edge implements Comparable<Edge> {
   public final int v, int w;
   public final double weight;

   public Edge(int v, int w, double weight) {
      this.v = v;
      this.w = w;
      this.weight = weight;
   }

   public int other(int vertex) {
      if (vertex == v) return w;
      else return v;
   }

   public int compareTo(Edge f) {
      Edge e = this;
      if      (e.weight < f.weight) return -1;
      else if (e.weight > f.weight) return +1;
      else                          return  0;
  }
}
```


Weighted Graph: Java Implementation

Identical to `Graph.java` but use `Edge` adjacency lists instead of `int`.

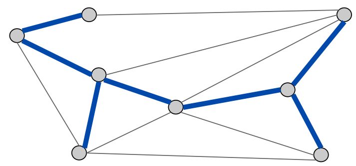
```
public class WeightedGraph {
   private int V;                 // # vertices
   private Sequence<Edge>[] adj;  // adjacency lists

   public Graph(int V) {
      this.V = V;
      adj = (Sequence<Edge>[]) new Sequence[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Sequence<Edge>();
   }

   public void insert(Edge e) {
      int v = e.v, w = e.w;
      adj[v].add(e);
      adj[w].add(e);
   }

   public Iterable<Edge> adj(int v) { return adj[v]; }

}
```

MST Structure

Spanning Tree

MST. Given connected graph G with positive edge weights, find a min weight set of edges that connects all of the vertices.

Def. A spanning tree of a graph G is a subgraph T that is connected and acyclic.



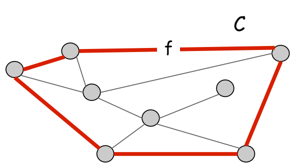
Property. MST of G is always a spanning tree.

Greedy Algorithms

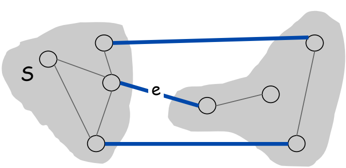
Simplifying assumption. All edge costs c_e are distinct.

Cycle property. Let C be any cycle, and let f be the max cost edge belonging to C. Then the MST does not contain f.

Cut property. Let S be any subset of vertices, and let e be the min cost edge with exactly one endpoint in S. Then the MST contains e.



f is not in the MST



e is in the MST

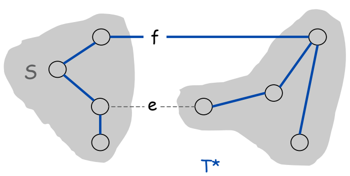
Cycle Property

Simplifying assumption. All edge costs c_e are distinct.

Cycle property. Let C be any cycle in G, and let f be the max cost edge belonging to C. Then the MST T* does not contain f.

Pf. [by contradiction]

- Suppose f belongs to T*. Let's see what happens.
- Deleting f from T* disconnects T*. Let S be one side of the cut.
- Some other edge in C, say e, has exactly one endpoint in S.
- $T = T^* \cup \{ e \} - \{ f \}$ is also a spanning tree.
- Since $c_e < c_f$, cost(T) < cost(T*).
- This is a contradiction. ▪



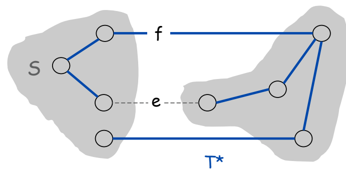
Cut Property

Simplifying assumption. All edge costs c_e are distinct.

Cut property. Let S be any subset of vertices, and let e be the min cost edge with exactly one endpoint in S. Then the MST T* contains e.

Pf. [by contradiction]

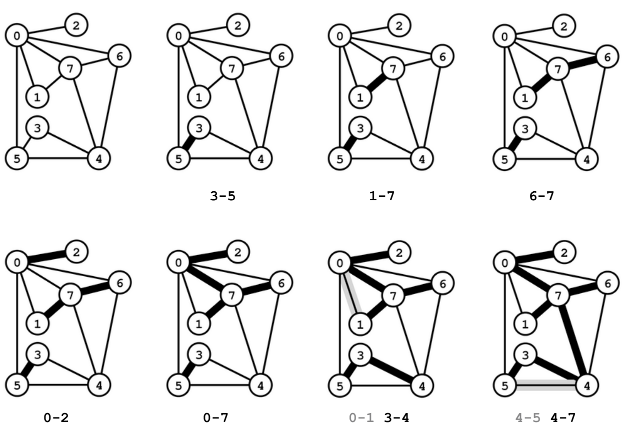
- Suppose e does not belong to T*. Let's see what happens.
- Adding e to T* creates a (unique) cycle C in T*.
- Some other edge in C, say f, has exactly one endpoint in S.
- T = T* ∪ { e } – { f } is also a spanning tree.
- Since $c_e < c_f$, cost(T) < cost(T*).
- This is a contradiction. ▪



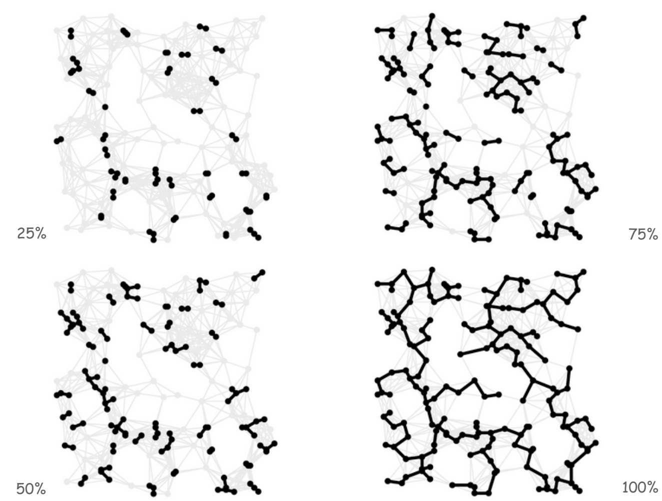
Kruskal's Algorithm

Kruskal's Algorithm: Example

Kruskal's algorithm. [Kruskal, 1956] Consider edges in ascending order of cost. Add the next edge to T unless doing so would create a cycle.



Kruskal's Algorithm: Example

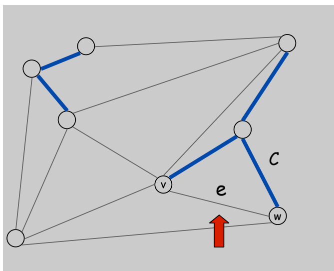


Kruskal's Algorithm: Proof of Correctness

Theorem. Kruskal's algorithm computes the MST.

why?

Pf. [case 1] If adding e to T creates a cycle C, then e is the max weight edge in C. The cycle property asserts that e is not in the MST.

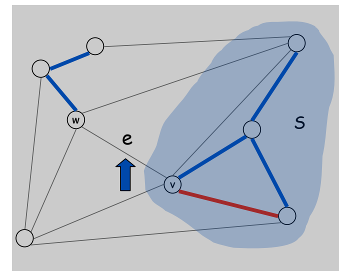


Kruskal's Algorithm: Proof of Correctness

Theorem. Kruskal's algorithm computes the MST.

Pf. [case 2] If adding e = (v, w) to T does not create a cycle, then e is the min weight edge with exactly one endpoint in S, so the cut property asserts that e is in the MST. ▪

set of vertices in v's connected component



Kruskal's Algorithm: Implementation

Q. How to check if adding an edge to T would create a cycle?

A1. Naïve solution: use DFS.

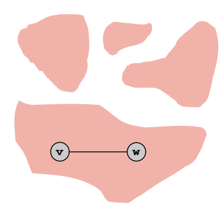
- O(V) time per cycle check.
- O(E V) time overall.

Kruskal's Algorithm: Implementation

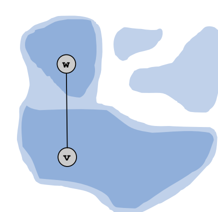
Q. How to check if adding an edge to T would create a cycle?

A2. Use the union-find data structure.

- Maintain a set for each connected component.
- If v and w are in same component, then adding v-w creates a cycle.
- To add v-w to T, merge sets containing v and w.



Case 1: adding v-w creates a cycle



Case 2: add v-w to T and merge sets

Kruskal's Algorithm: Java Implementation

```java
public class Kruskal {
   private Sequence<Edge> mst = new Sequence<Edge>();

   public Kruskal(WeightedGraph G) {
      // sort edges in ascending order
      Edge[] edges = G.edges();
      Arrays.sort(edges);

      // greedily add edges to MST
      UnionFind uf = new UnionFind(G.V());
      for (int i = 0; (i < E) && (mst.size() < G.V()-1); i++) {
         int v = edges[i].v;
         int w = edges[i].w;
         if (!uf.find(v, w)) {
            uf.unite(v, w);
            mst.add(edges[i]);
         }
      }
   }

   public Iterable<Edge> mst() { return mst; }
}
```

safe to stop early if tree already has V-1 edges

Kruskal's Algorithm: Running Time

Kruskal running time. O(E log V).

$E \le V^2$ so O(log E) is O(log V)

Operation	Frequency	Time per op
sort	1	E log V
union	V	log* V †
find	E	log* V †

† amortized bound using weighted quick union with path compression

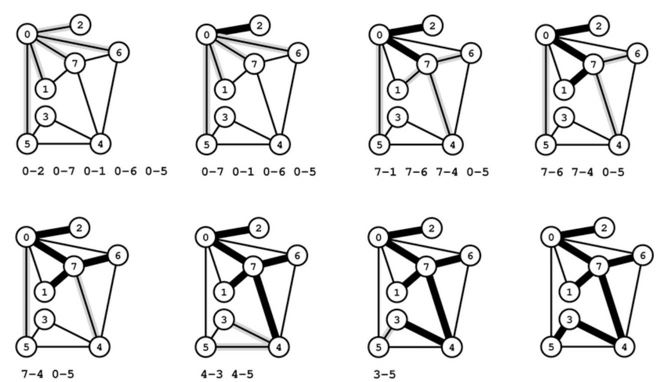
Remark. If edges already sorted: O(E log* V) time.

recall: log* V ≤ 5 in this universe

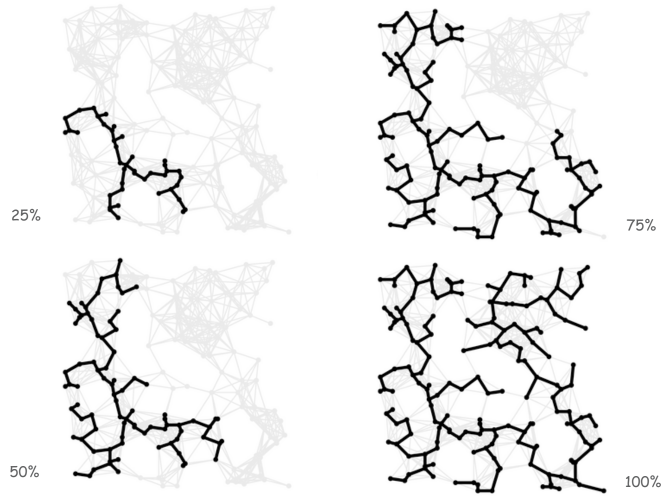
Prim's Algorithm

Prim's Algorithm: Example

Prim's algorithm. [Jarník 1930, Dijkstra 1957, Prim 1959]
Start with vertex 0 and greedily grow tree T. At each step, add cheapest edge that has exactly one endpoint in T.



Prim's Algorithm: Example

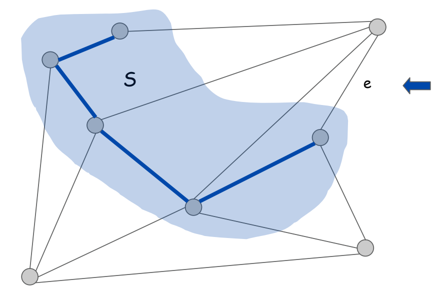


Prim's Algorithm: Proof of Correctness

Theorem. Prim's algorithm computes the MST.

Pf.

- Let S be the subset of vertices in current tree T.
- Prim adds the cheapest edge e with exactly one endpoint in S.
- Cut property asserts that e is in the MST. ▪



Prim's Algorithm: Implementation

Q. How to find cheapest edge with exactly one endpoint in S?

A1. Brute force: try all edges.

- O(E) time per spanning tree edge.
- O(E V) time overall.

Prim's Algorithm: Implementation

Q. How to find cheapest edge with exactly one endpoint in S?

A2. Maintain edges with (at least) one endpoint in S in a priority queue.

- Delete min to determine next edge e to add to T.
- Disregard e if both endpoints are in S.
- Upon adding e to T, add to PQ the edges incident to one endpoint.

the one not already in S

Running time.

- O(log V) time per edge (using a binary heap).
- O(E log V) time overall.

Prim's Algorithm: Java Implementation

```
public class LazyPrim {
   private Sequence<Edge> mst = new Sequence<Edge>();

   public LazyPrim(WeightedGraph G) {
      boolean[] marked = new boolean[G.V()];
      MinPQ<Edge> pq = new MinPQ<Edge>();
      int s = 0;
      marked[s] = true;
      for (Edge e : G.adj(0)) pq.insert(s);

      while (!pq.isEmpty()) {
         Edge e = pq.delMin();
         int v = e.v, w = e.w;
         if (!marked[v] || !marked[w]) mst.add(e);
         if (!marked[v])
            for (Edge f : G.adj(v)) pq.insert(f);
         if (!marked[w])
            for (Edge f : G.adj(w)) pq.insert(f);
         marked[v] = marked[w] = true;
      }
   }
}
```

is v in S?

add all edges incident to s

add edge to MST unless both endpoints are already in S

these edges have exactly one endpoint in S

Removing the Distinct Edge Costs Assumption

Simplifying assumption. All edge costs c_e are distinct.

One way to remove assumption. Kruskal and Prim only access edge weights through `compareTo()`; suffices to introduce tie-breaking rule.

```
public int compareTo(Edge f) {
   Edge e = this;
   if (e.weight < f.weight) return -1;
   if (e.weight > f.weight) return +1;
   if (e.v < f.v) return -1;
   if (e.v > f.v) return +1;
   if (e.w < f.w) return -1;
   if (e.w > f.w) return +1;
   return  0;
}
```


Removing the Distinct Edge Costs Assumption

Simplifying assumption. All edge costs c_e are distinct.

Fact. Prim and Kruskal don't actually rely on the assumption.

only our proof of correctness does!

Advanced MST Algorithms

Advanced MST Algorithms

Year	Worst Case	Discovered By
1975	E log log V	Yao
1976	E log log V	Cheriton-Tarjan
1984	E log* V, E + V log V	Fredman-Tarjan
1986	E log (log* V)	Gabow-Galil-Spencer-Tarjan
1997	E α(V) log α(V)	Chazelle
2000	E α(V)	Chazelle
2002	optimal	Pettie-Ramachandran
20xx	E	???

deterministic comparison based MST algorithms

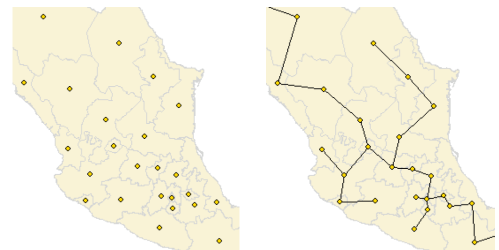
Year	Problem	Time	Discovered By
1976	Planar MST	E	Cheriton-Tarjan
1992	MST Verification	E	Dixon-Rauch-Tarjan
1995	Randomized MST	E	Karger-Klein-Tarjan

related problems

Euclidean MST

Euclidean MST. Given N points in the plane, find MST connecting them.

- Distances between point pairs are Euclidean distances.



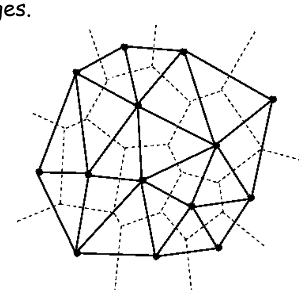
Brute force. Compute $\Theta(N^2)$ distances and run Prim's algorithm.
Ingenuity. Exploit geometry and do it in O(N log N).

Euclidean MST

Key geometric fact. Edges of the Euclidean MST are edges of the Delaunay triangulation.

Euclidean MST algorithm.

- Compute Voronoi diagram to get Delaunay triangulation.
- Run Kruskal's MST algorithm on Delaunay edges.



Running time. O(N log N).

- Fact: $\leq$ 3N Delaunay edges since it's planar.
- O(N log N) for Voronoi.
- O(N log N) for Kruskal.

Lower bound. Any comparison-based Euclidean MST algorithm requires Ω(N log N) comparisons.