Combinatorial Search

Robert Sedgewick and Kevin Wayne • Copyright © 2006 • http://www.Princeton.EDU/~cos226

Overview

Exhaustive search. Iterate through all elements of a search space.

Backtracking. Systematic method for generating all solutions to a problem, by successively augmenting partial solutions.

Applicability. Huge range of problems (include NP-hard ones).

Caveat. Search space is typically exponential in size ⇒ effectiveness is limited to relatively small instances.

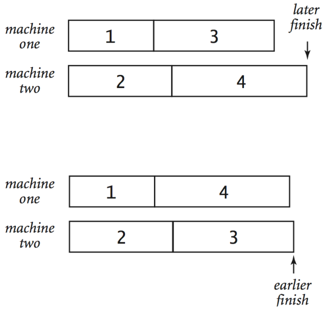
Enumerating Subsets

Scheduling

Scheduling (set partitioning). Given n jobs of varying length, divide among two machines to minimize the time the last job finishes.

or, equivalently, difference between finish times

job	length
1	1.41
2	1.73
3	2.00
4	2.23



Remark. NP-hard.

Enumerating Subsets: Natural Binary Code

Enumerating subsets. Given n items, enumerate all 2^n subsets.

- Count in binary from 0 to 2^n - 1.
- Look at binary representation.

integer	*binary code*	*machine one*	*machine two*
0	0 0 0 0	empty	4 3 2 1
1	0 0 0 1	1	4 3 2
2	0 0 1 0	2	4 3 1
3	0 0 1 1	2 1	4 3
4	0 1 0 0	3	4 2 1
5	0 1 0 1	3 1	4 2
6	0 1 1 0	3 2	4 1
7	0 1 1 1	3 2 1	4
8	1 0 0 0	4	3 2 1
9	1 0 0 1	4 1	3 2
10	1 0 1 0	4 2	3 1
11	1 0 1 1	4 2 1	3
12	1 1 0 0	4 3	2 1
13	1 1 0 1	4 3 1	2
14	1 1 1 0	4 3 2	1
15	1 1 1 1	4 3 2 1	empty

Enumerating Subsets: Natural Binary Code

Enumerating subsets. Given n items, enumerate all 2^n subsets.

- Count in binary from 0 to 2^n - 1.
- Look at binary representation.

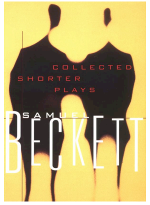
```java
long N = 1 << n;
for (long i = 0; i < N; i++) {
   for (int bit = 0; bit < n; bit++) {
      if (((i >> bit) & 1) == 1)
         System.out.print(bit + " ");
   }
   System.out.println();
}
```

Samuel Beckett

Quad. Starting with empty stage, 4 characters enter and exit one at a time, such that each subset of actors appears exactly once.

code	*subset*	*move*
0 0 0 0	*empty*	
0 0 0 1	1	enter 1
0 0 1 1	2 1	enter 2
0 0 1 0	2	exit 1
0 1 1 0	3 2	enter 3
0 1 1 1	3 2 1	enter 1
0 1 0 1	3 1	exit 2
0 1 0 0	3	exit 1
1 1 0 0	4 3	enter 4
1 1 0 1	4 3 1	enter 1
1 1 1 1	4 3 2 1	enter 2
1 1 1 0	4 3 2	exit 1
1 0 1 0	4 2	exit 3
1 0 1 1	4 2 1	enter 1
1 0 0 1	4 1	exit 2
1 0 0 0	4	exit 1

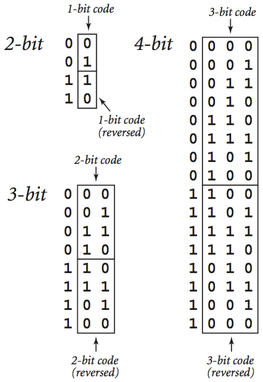
ruler function



Enumerating Subsets: Binary Reflected Gray Code

Binary reflected Gray code. The n-bit code is:

- the (n-1) bit code with a 0 prepended to each word, followed by
- the (n-1) bit code in reverse order, with a 1 prepended to each word.



2-bit

```
0 0
0 1
1 1
1 0
```

3-bit

```
0 0 0
0 0 1
0 1 1
0 1 0
1 1 0
1 1 1
1 0 1
1 0 0
```

4-bit

```
0 0 0 0
0 0 0 1
0 0 1 1
0 0 1 0
0 1 1 0
0 1 1 1
0 1 0 1
0 1 0 0
1 1 0 0
1 1 0 1
1 1 1 1
1 1 1 0
1 0 1 0
1 0 1 1
1 0 0 1
1 0 0 0
```

Beckett: Java Implementation

```
public static void moves(int n, boolean enter) {
   if (n == 0) return;
   moves(n-1, true);
   if (enter) System.out.println("enter " + n);
   else       System.out.println("exit  " + n);
   moves(n-1, false);
}
```

```
% java Beckett 4
enter 1
enter 2
exit  1
enter 3
enter 1
exit  2
exit  1
enter 4
enter 1
enter 2
exit  1
exit  3
enter 1
exit  2
exit  1
```

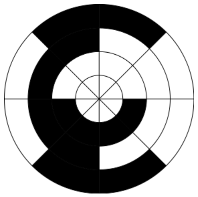
stage directions for 3-actor play

`moves(3, true)`

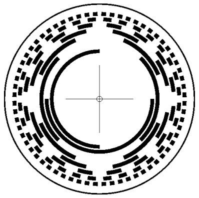
reverse stage directions for 3-actor play

`moves(3, false)`

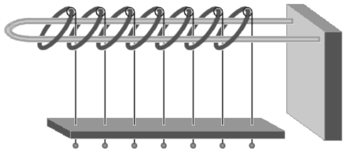
More Applications of Gray Codes



3-bit rotary encoder



8-bit rotary encoder



Chinese ring puzzle

Scheduling (using Gray Code)

gap = sum

+2.23 if job 4 on machine one
-2.23 if job 4 on machine two

	a[0]	a[1]	a[2]	a[3]	a[4]
	7.37	1.41	1.73	2.00	2.23
1	4.55	$\boxed{-1.41}$	1.73	2.00	2.23
2	1.09	-1.41	$\boxed{-1.73}$	2.00	2.23
1	3.91	$\boxed{1.41}$	-1.73	2.00	2.23
3	-0.09	1.41	-1.73	$\boxed{-2.00}$	2.23
1	-2.91	$\boxed{-1.41}$	-1.73	-2.00	2.23
2	0.55	-1.41	$\boxed{1.73}$	-2.00	2.23
1	3.38	$\boxed{1.41}$	1.73	-2.00	2.23
4	-1.08	1.41	1.73	-2.00	$\boxed{-2.23}$
1	-3.91	$\boxed{-1.41}$	1.73	-2.00	-2.23
2	-7.37	-1.41	$\boxed{-1.73}$	-2.00	-2.23
1	-4.55	$\boxed{1.41}$	-1.73	-2.00	-2.23
3	-0.55	1.41	-1.73	$\boxed{2.00}$	-2.23
1	-3.38	$\boxed{-1.41}$	-1.73	2.00	-2.23
2	0.09	-1.41	$\boxed{1.73}$	2.00	-2.23
1	2.91	$\boxed{1.41}$	1.73	2.00	-2.23

Beckett's stage directions

flip job 4 from machine one to machine two

Scheduling: Java Implementation

current schedule

best schedule so far

```
public static void moves(int n, double[] a, double[] b) {
   if (n == 0) return;
   moves(n-1, a, b);

   a[n] = -a[n];
   a[0] += 2*a[n];
   if (Math.abs(a[0]) < Math.abs(b[0]))
      for (int i = 0; i < a.length; i++)
         b[i] = a[i];

   moves(n-1, a, b);
}
```

flip machine for job n ; check schedule

sum

job lengths

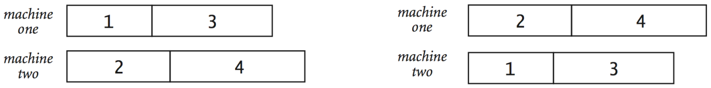
```
int[] a  = { 7.37, 1.41, 1.73, 2.00, 2.23 };

int[] b  = { 7.37, 1.41, 1.73, 2.00, 2.23 };
```

best schedule so far

Exploiting Symmetry

Exploit symmetry.

- Half of schedules are redundant.



- Fix job n on machine one ⇒ twice as fast.

Space-Time Tradeoff

job	length
1	1.41
2	1.73
3	2.00
4	2.23
5	3.00
6	0.35

Space-time tradeoff.

- Enumerate all subsets of first n/2 jobs; sort by gap.

gap (subset)	-5.14 (empty)	-2.32 (1)	-1.68 (2)	-1.14 (3)	1.14 (1 2)	1.68 (1 3)	2.32 (2 3)	5.14 (1 2 3)

- Enumerate all subsets of last n/2 jobs; for each subset, binary search to find for best matching subset among first n/2 jobs.

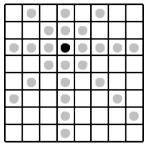
gap (subset)	-5.58 (empty)	-1.12 (4)	0.42 (5)	-4.88 (6)	4.48 (4 5)	-0.42 (4 6)	1.12 (5 6)	5.58 (4 5 6)
best match	5.14 (1 2 3)	1.14 (1 2)	-1.14 (3)	5.14 (1 2 3)	-5.14 (empty)	1.14 (2)	-1.14 (3)	-5.14 (empty)
	-0.44 (1 2 3)	0.02 (1 2 4)	-0.72 (3 5)	0.26 (1 2 3 6)	-0.26 (4 5)	0.72 (2 4 6)	0.02 (3 5 6)	0.44 (1 2 3)

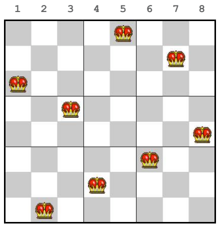
- Reduces running time from 2^n to $2^{n/2} \log n$ by consuming $2^{n/2}$ memory.

Enumerating Permutations

8-Queens Problem

8-queens problem. Place 8 queens on a chessboard so that no queen can attack any other queen.





Representation. Can represent solution as a permutation: `q[i]` = column of queen in row `i`.

```
int[] q = { 5, 7, 1, 3, 8, 6, 4, 2 };
```

queens `i` and `j` can attack each other if `|q[i] + i| = |q[j] + j|`

Enumerating Permutations

Permutations. Given n items, enumerate all n! permutations.

order matters

3-element permutations

```
1 2 3
1 3 2
2 1 3
2 3 1
3 1 2
3 2 1
```

4-element permutations

```
1 | 2 3 4     2 | 1 3 4     3 | 1 2 4     3 | 1 2 4
1 | 2 4 3     2 | 1 4 3     3 | 1 4 2     3 | 1 4 2
1 | 3 2 4     2 | 3 1 4     3 | 2 1 4     3 | 2 1 4
1 | 3 4 2     2 | 3 4 1     3 | 2 4 1     3 | 2 4 1
1 | 4 2 3     2 | 4 1 3     3 | 4 1 2     3 | 4 1 2
1 | 4 3 2     2 | 4 3 1     3 | 4 2 1     3 | 4 2 1
```

1 followed by any permutation of 2 3 4

2 followed by any permutation of 1 3 4

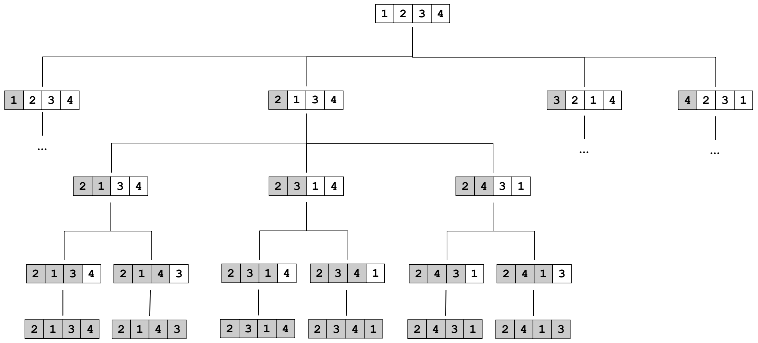
3 followed by any permutation of 1 2 4

3 followed by any permutation of 1 2 4

Enumerating all Permutations

To enumerate all permutations of a set of n elements:

- For each element a_i
  - put a_i first, then append
  - a permutation of the remaining elements ($a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n-1}$)



Enumerating All Permutations: Java Implementation

permutations of a[n], …, a[N-1]

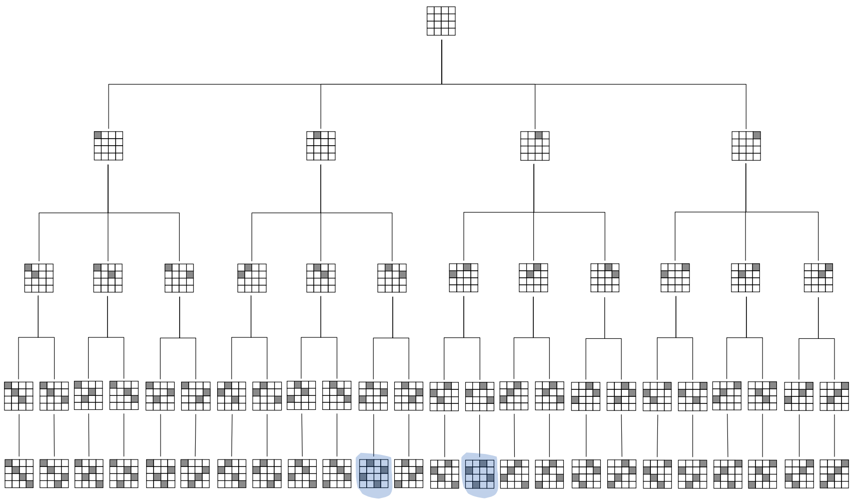
```
private static void enumerate(int[] a, int n) {
   int N = a.length;
   if (n == N) printPermutations(a);
   for (int i = n; i < N; i++) {
      swap(q, i, n);
      enumerate(a, n+1);
      swap(q, n, i);
   }
}
```

cleans up after itself

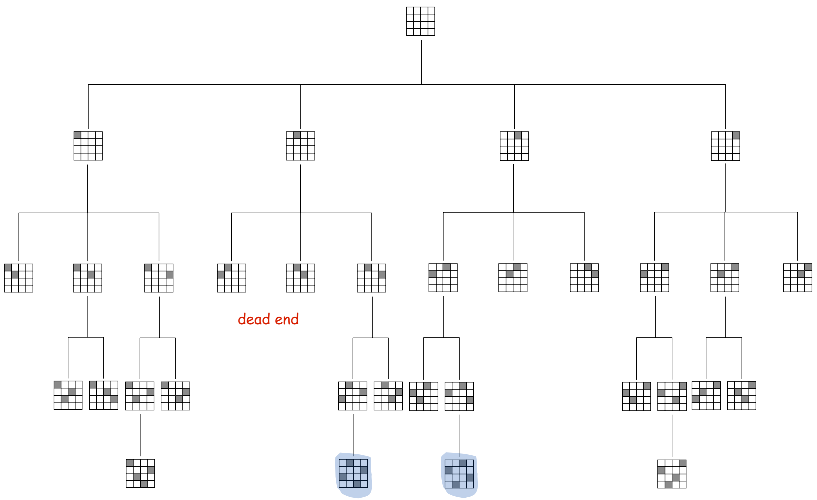
```
int N = 4;
int[] a = { 1, 2, 3, 4 };
enumerate(a, N);
```

Pruning

4-Queens Search Tree



4-Queens Search Tree (pruned)



N-Queens: Backtracking Solution

```
private static void enumerate(int[] q, int n) {
   int N = q.length;
   if (n == N) printQueens(q);
   for (int i = n; i < N; i++) {
      swap(q, i, n);
      if (isConsistent(q, n)) enumerate(q, n+1);
      swap(q, n, i);
   }
}
```

stop enumerating if adding the n^{th} queen leads to a violation

```
int N = 4;
int[] q = { 1, 2, 3, 4 };
enumerate(q, N);
```

Sudoku

Sudoku

Sudoku. Fill 9-by-9 grid so that every row, column, and box contains the digits 1 through 9.

7		8				3		
			2		1			
5								
	4						2	6
3				8				
			1				9	
	9		6					4
				7		5		

Remark. Natural generalization is NP-hard.

Sudoku

Sudoku. Fill 9-by-9 grid so that every row, column, and box contains the digits 1 through 9.

7	2	8	9	4	6	3	1	5
9	3	4	2	5	1	6	7	8
5	1	6	7	3	8	2	4	9
1	4	7	5	9	3	8	2	6
3	6	9	4	8	2	1	5	7
8	5	2	1	6	7	4	9	3
2	9	3	6	1	5	7	8	4
4	8	1	3	7	9	5	6	2
6	7	5	8	2	4	9	3	1

Remark. Natural generalization is NP-hard.

Sudoku

Linearize. Treat 9-by-9 array as an array of length 81.

7		8				3		
			2		1			
5								
	4						2	6
3				8				
			1				9	
	9		6					4
				7		5		

7		8					3		...	
0	1	2	3	4	5	6	7	8		80

Enumerate all assignments. Count from 0 to 9^{81} - 1 in base 9.

using digits 1 to 9

Sudoku: Backtracking Solution

Backtracking. Iterate through elements of search space.

- For each empty cell, there are 9 possible choices.
- Make one choice and recur.
- If you reach a contradiction, go back to previous choice, and make next available choice.

Pruning. Stop as soon as you reach a contradiction.

Improvements.

- Choose most constrained cell to examine next.
- Knuth's "dancing links."

Sudoku: Java Implementation

```
private static void solve(int[] board, int cell) {

   // found the solution
   if (cell == 81) { show(board); return; }

   // skip over cell n since it has fixed value
   if (board[cell] != 0) { solve(board, cell + 1); return; }

   // try all 9 possibilities
   for (int n = 1; n <= 9; n++) {
      if (isConsistent(board, cell, n)) {
         board[cell] = n;
         solve(board, cell + 1);
      }
   }
   board[cell] = 0;
}
```

don't bother if a Sudoku constraint is already violated

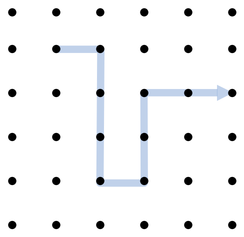
cleans up after itself

```
int[] board = { 7, 0, 8, 0, 0, 0, 3, … };
solve(board, 0);
```

Enumerating all Paths in a Grid

All Paths on a Grid

All paths. Enumerate all simple paths on a grid of adjacent sites.

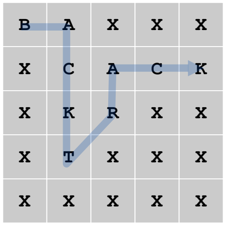


Application. Self-avoiding lattice walk to model polymer chains.

no atoms can occupy same position at same time

Boggle

Boggle. Find all words that can be formed by tracing a simple path of adjacent cubes (left, right, up, down, diagonal).



B	A	X	X	X
X	C	A	C	K
X	K	R	X	X
X	T	X	X	X
X	X	X	X	X

Pruning. Stop as soon as no word in dictionary contains string of letters on current path as a prefix ⇒ use a trie.

```
B
BA
BAX
```

Boggle: Java Implementation

```
// find all words starting at (i, j)
private void dfs(String prefix, int i, int j) {
   if (i < 0 || i >= N) return;
   if (j < 0 || j >= N) return;
   if (visited[i][j])    return;

   if (!dictionary.containsAsPrefix(prefix)) return;

   visited[i][j] = true;
   prefix = prefix + board[i][j];

   if (dictionary.contains(prefix))
      found.add(prefix);

   // recur on all 8 neighbors
   for (int ii = -1; ii <= 1; ii++)
      for (int jj = -1; jj <= 1; jj++)
         dfs(prefix, i + ii, j + jj);

   visited[i][j] = false;
}
```

out-of-bounds or self-intersecting

don't bother continuing if no possible words

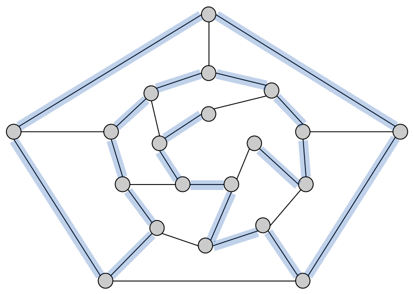
add to set of found words

backtrack

Enumerating all Paths in a Graph

Hamilton Path

Hamilton path. Find a simple path that visits every vertex exactly once.

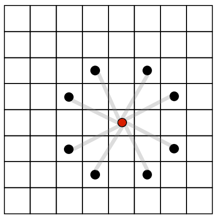


Remark. Euler path easy, but Hamilton path is NP-complete.

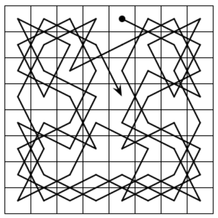
visit every edge exactly once

Knight's Tour

Knight's tour. Find a sequence of moves for a knight so that, starting from any square, it visits every square on a chessboard exactly once.



legal knight moves



a knight's tour

Solution. Find a Hamilton path in knight's graph.

Hamilton Path: Backtracking Solution

Backtracking solution. To find Hamilton path starting at `v`:

- Add `v` to current path.
- For each vertex `w` adjacent to `v`
  - find a simple path starting at `w` using all remaining vertices
- Remove `v` from current path.

How to implement?

- To keep track of path: use a stack.
- To record which vertices are on the path: use a boolean array.
- To recursively visit vertices: use depth-first search.

Heuristic. Choose vertex with fewest unvisited neighbors.

Hamilton Path: Java Implementation

```
public class HamiltonPath {
   private boolean[] onPath;
   private Stack<Integer> path = new Stack<Integer>();

   public HamiltonPath(Graph G) {
      onPath = new boolean[G.V()];
      for (int v = 0; v < G.V(); v++)
         dfs(G, v);
   }

   private void dfs(Digraph G, int v) {
      path.push(v);
      onPath[v] = true;

      if (path.size() == G.V()) System.out.println(path);

      for (int w : G.adj(v))
         if (!onPath[w]) dfs(G, w);

      path.pop();
      onPath[v] = false;
   }
}
```

add v to the current path

don't bother further exploration
if w is already on the current path

remove v from the current path

The Longest Path

Recorded by Dan Barrett in 1988 while a student at Johns Hopkins during a difficult algorithms final.

Woh-oh-oh-oh, find the longest path!
Woh-oh-oh-oh, find the longest path!

If you said P is NP tonight,
There would still be papers left to write,
I have a weakness,
I'm addicted to completeness,
And I keep searching for the longest path.

The algorithm I would like to see
Is of polynomial degree,
But it's elusive:
Nobody has found conclusive
Evidence that we can find a longest path.

I have been hard working for so long.
I swear it's right, and he marks it wrong.
Some how I'll feel sorry when it's done: GPA 2.1
Is more than I hope for.

Garey, Johnson, Karp and other men (and women)
Tried to make it order N log N.
Am I a mad fool
If I spend my life in grad school,
Forever following the longest path?

Woh-oh-oh-oh, find the longest path!
Woh-oh-oh-oh, find the longest path!
Woh-oh-oh-oh, find the longest path.