Semantics for Pictures

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1 Introduction

In this document we will describe a simple language for describing pictures. We will describe our language by providing a precise denotational semantics. However, we will begin by introducing some key concepts in an informal way.

1.1 Concepts

Point - A location on the plane of real numbers ($\mathbb{R}^2$)

Shape - A set of points.

Color - A color is a mixture of the primary colors red, green, and blue.

Texture - An assignment of a color to every point on the plane.

Layer - A layer is a partial assignment of colors to points on the plane. Layers are built by combining a shape with a texture. A point is assigned a color if the point is a member of a particular shape. The color the point has is derived from the texture.

Picture - A picture is an ordered collection of layers that results in a partial assignment of colors to points. If two layers in a picture both assign a color to a point we choose the color of the top most layer.

Image - An assignment of a colors to every point on the plane. An image can be created from a picture by assigning a default value to all points in a picture that do not have a value assigned.

Notice none of our definitions refer to pixles or resolutions. We wish to describe the esence of a picture without specifying how one would actually display an image on a device with a limited resolution.

2 A Language for Describing Pictures

2.1 Syntax

$$
\begin{array}{rrcl}
\textit{real numbers} & x, y & ::= & \ldots \\
\textit{scale factors} & a, b & ::= & x \text{ where } x \neq 0 \\
\textit{points} & p & ::= & (x, y) \\
\textit{shape} & s & ::= & \mathsf{everything} \mid \mathsf{nothing} \\
 & & \mid & \mathsf{union}(s_1, s_2) \mid \mathsf{intersect}(s_1, s_2) \mid \mathsf{difference}(s_1, s_2) \\
 & & \mid & \mathsf{ellipse}(p, a, b) \mid \mathsf{halfplane}(p_0, p_1) \\
 & & \mid & \mathsf{translate}(p, s) \mid \mathsf{scaleXY}(a, b, s) \\
\textit{color} & c & ::= & \mathsf{red} \mid \mathsf{green} \mid \mathsf{blue} \mid \ldots \\
\textit{texture} & t & ::= & \mathsf{constant}(c) \mid \ldots \\
\textit{picture} & \textit{pict} & ::= & \mathsf{layer}(s, t) \mid \textit{pict}_1 \triangleright \textit{pict}_2 \\
\textit{image} & \textit{image} & ::= & (\textit{pict}, c)
\end{array}
$$

2.2 Semantics

$$
\begin{array}{rcl}
\mathcal{S}[\![\mathsf{everything}]\!] & \cong & \mathbb{R}^2 \\
\mathcal{S}[\![\mathsf{nothing}]\!] & \cong & \{\} \\
\mathcal{S}[\![\mathsf{union}(s_1, s_2)]\!] & \cong & \{p \mid p \in \mathcal{S}[\![s_1]\!] \ \vee \ p \in \mathcal{S}[\![s_2]\!]\} \\
\mathcal{S}[\![\mathsf{intersect}(s_1, s_2)]\!] & \cong & \{p \mid p \in \mathcal{S}[\![s_1]\!] \ \wedge \ p \in \mathcal{S}[\![s_2]\!]\} \\
\mathcal{S}[\![\mathsf{difference}(s_1, s_2)]\!] & \cong & \{p \mid p \in \mathcal{S}[\![s_1]\!] \ \wedge \ p \notin \mathcal{S}[\![s_2]\!]\} \\
\mathcal{S}[\![\mathsf{ellipse}((x_0, y_0), a, b)]\!] & \cong & \{(x, y) \mid (x - x_0)^2/a^2 + (y - y_0)^2/b^2 \leq 1\} \\
\mathcal{S}[\![\mathsf{halfplane}((x_0, y_0), (x_1, y_1))]\!] & \cong & \{(x, y) \mid (y - y_0)(x_1 - x_0) \geq (x - x_0)(y_1 - y_0)\} \\
\mathcal{S}[\![\mathsf{translate}((x_0, y_0), s)]\!] & \cong & \{(x, y) \mid \\
 & & \quad \exists x_s, y_s.\ x = x_s + x_0 \ \wedge \ y = y_s + y_0 \ \wedge \ (x_s, y_s) \in \mathcal{S}[\![s]\!]\} \\
\mathcal{S}[\![\mathsf{scaleXY}(a, b, s)]\!] & \cong & \{(x, y) \mid \\
 & & \quad \exists x_s, y_s\ x = a x_s \ \wedge \ y = b y_s \ \wedge \ (x_s, y_s) \in \mathcal{S}[\![s]\!]\}
\end{array}
$$

$$
\mathcal{T}[\![\mathsf{constant}(c_0)]\!] \ \cong \ \{(p, c) \mid c = c_0\}
$$

$$
\begin{array}{rcl}
\mathcal{P}[\![\mathsf{layer}(s, t)]\!] & \cong & \{(p, c) \mid p \in \mathcal{S}[\![s]\!] \ \wedge \ (p, c) \in \mathcal{T}[\![t]\!]\} \\
\mathcal{P}[\![pict_1 \triangleright pict_2]\!] & \cong & \{(p, c) \mid (p, c) \in \mathcal{P}[\![pict_1]\!] \\
 & & \qquad \vee \ (\neg(\exists c'.(p, c') \in \mathcal{P}[\![pict_1]\!]) \ \wedge \ (p, c) \in \mathcal{P}[\![pict_2]\!])\}
\end{array}
$$

$$
\begin{array}{rcl}
\mathcal{I}[\![(pict, c)]\!] & \cong & \{(p, c_0) \mid (p, c) \in \mathcal{P}[\![pict]\!] \\
 & & \qquad \vee \ (\neg(\exists c'.(p, c') \in \mathcal{P}[\![pict_1]\!]) \ \wedge \ c = c_0)\}
\end{array}
$$

3 Theorems About Shapes

3.1 Some Well Known Shapes

3.1.1 The Unit Circle

The unit circle centered at the origin is defined by the set

$$\{(x, y) \mid x^2 + y^2 \leq 1\}$$

It is easy to show that $\mathcal{S}[\![\mathsf{ellipse}((0, 0), 1, 1)]\!]$ is the unit circle

$$
\begin{array}{rcl}
\mathcal{S}[\![\mathsf{ellipse}((0, 0), 1, 1)]\!] & \cong & \{(x, y) \mid (x - 0)^2/1^2 + (y - 0)^2/1^2 \leq 1\} \\
 & \cong & \{(x, y) \mid x^2 + y^2 \leq 1\}
\end{array}
$$

3.1.2 A Semi-Circle

A semi-circle lying in the non-negative y-quadrant centered at the origin is defined by the set

$$\{(x, y) \mid x^2 + y^2 \leq 1 \ \wedge \ y \geq 0\}$$

We will show that

$$\mathcal{S}[\![\mathsf{intersect}(\mathsf{halfplane}((0,0),(1,0)), \mathsf{ellipse}((0,0),1,1))]\!]$$

is such a semi-circle. From our defintion we have

$$\begin{aligned}&\mathcal{S}[\![\mathsf{intersect}(\mathsf{halfplane}((0,0),(1,0)), \mathsf{ellipse}((0,0),1,1))]\!] \cong \\ &\quad \{p \mid p \in \mathcal{S}[\![\mathsf{halfplane}((0,0),(1,0))]\!] \ \wedge \ p \in \mathcal{S}[\![\mathsf{ellipse}((0,0),1,1))]\!]\}\end{aligned}$$

from our previous result about the unit circle we know.

$$\begin{aligned}&\{p \mid p \in \mathcal{S}[\![\mathsf{halfplane}((0,0),(1,0))]\!] \ \wedge \ p \in \mathcal{S}[\![\mathsf{ellipse}((0,0),1,1)]\!]\} \cong \\ &\quad \{p \mid p \in \mathcal{S}[\![\mathsf{halfplane}((0,0),(1,0))]\!] \ \wedge \ p \in \{(x, y) \mid x^2 + y^2 \leq 1\}\}\end{aligned}$$

Again from our defintion we have

$$\{p \mid p \in \{(x, y) \mid (y - 0)(1 - 0) \geq (x - x_0)(0 - 0)\} \ \wedge \ p \in \{(x, y) \mid x^2 + y^2 \leq 1\}\}$$

by a simple change of variables we have

$$\{(x, y) \mid (x, y) \in \{(x, y) \mid (y - 0)(1 - 0) \geq (x - 0)(0 - 0)\} \ \wedge \ (x, y) \in \{(x, y) \mid x^2 + y^2 \leq 1\}\}$$

which is the same as

$$\{(x, y) \mid (y - 0)(1 - 0) \geq (x - 0)(0 - 0) \ \wedge \ x^2 + y^2 \leq 1\}$$

Simplifying we have

$$\{(x, y) \mid y \geq 0 \ \wedge \ x^2 + y^2 \leq 1\}$$

which is equivalent to the semi-circle

$$\{(x, y) \mid x^2 + y^2 \leq 1 \ \wedge \ y \geq 0\}$$

3.1.3 The Unit Square

The square centered at the origin with side length 1 is described by the set

$$\{(x, y) \mid -0.5 \leq x \leq 0.5 \ \wedge \ -0.5 \leq y \leq 0.5\}$$

The meaning of the following shape expression describes such a square

$$\begin{aligned}&\mathsf{insertsect}(\mathsf{halfplane}((-0.5, -0.5), (0.5, -0.5)), \\ &\quad \mathsf{insertsect}(\mathsf{halfplane}((0.5, -0.5), (0.5, 0.5)), \\ &\quad \mathsf{insertsect}(\mathsf{halfplane}((0.5, 0.5), (-0.5, 0.5)), \mathsf{halfplane}((-0.5, 0.5), (-0.5, -0.5)))))\end{aligned}$$

Laboriously expanding the above expression to its meaning gives us

$$\begin{array}{l}\{(x,y) \mid \\ \quad (y-(-0.5))(0.5-(-0.5)) \geq (x-(-0.5))((-0.5)-(-0.5)) \ \wedge \\ \quad (y-(-0.5))(0.5-0.5) \geq (x-0.5)(0.5-(-0.5)) \ \wedge \\ \quad (y-0.5)((-0.5)-0.5) \geq (x-0.5)(0.5-0.5) \ \wedge \\ \quad (y-0.5)((-0.5)-(-0.5)) \geq (x-(-0.5))((-0.5)-0.5)\}\end{array}$$

Simplifying we obtain

$$\begin{array}{l}\{(x,y) \mid \quad y+0.5 \geq 0 \ \wedge \\ \quad 0 \geq x-0.5 \ \wedge \\ \quad -y+0.5 \geq 0 \ \wedge \\ \quad 0 \geq -x-0.5\}\end{array}$$

Rearranging the above we have

$$\{(x,y) \mid 0 \geq x-0.5 \ \wedge \ 0 \geq -x-0.5 \ \wedge \ y+0.5 \geq 0 \ \wedge \ -y+0.5 \geq 0\}$$

Distributing terms we have

$$\{(x,y) \mid 0.5 \geq x \ \wedge \ x \geq -0.5 \ \wedge \ y \geq -0.5 \ \wedge \ 0.5 \geq y\}$$

We may rewrite the above as

$$\{(x,y) \mid x \leq 0.5 \ \wedge \ -0.5 \leq x \ \wedge \ -0.5 \leq y \ \wedge \ y \leq 0.5\}$$

which is logically equivalent to

$$\{(x,y) \mid -0.5 \leq x \leq 0.5 \ \wedge \ -0.5 \leq y \leq 0.5\}$$

3.2 Theorems About Scaling and Translation

It is easy to show the following equation hold

$$\begin{array}{rcl}\mathcal{S}[\![\mathsf{scaleXY}(a,b,\mathsf{nothing})]\!] & \cong & \mathsf{nothing} \\ \mathcal{S}[\![\mathsf{translate}(p,\mathsf{nothing})]\!] & \cong & \mathsf{nothing} \\ \mathcal{S}[\![\mathsf{scaleXY}(a,b,\mathsf{everything})]\!] & \cong & \mathsf{everything} \\ \mathcal{S}[\![\mathsf{translate}(p,\mathsf{everything})]\!] & \cong & \mathsf{everything}\end{array}$$

The first two are trivially true while the second two rely on the fact that $\mathbb{R}$ is closed under addition and multiplication. The following equations also hold

$$
\begin{aligned}
\mathcal{S}[\![\mathsf{scaleXY}(a,b,\mathsf{union}(s_1,s_2))]\!] &\cong \mathcal{S}[\![\mathsf{union}(\mathsf{scaleXY}(a,b,s_1),\mathsf{scaleXY}(a,b,s_2))]\!]\\
\mathcal{S}[\![\mathsf{scaleXY}(a,b,\mathsf{intersect}(s_1,s_2))]\!] &\cong \mathcal{S}[\![\mathsf{intersect}(\mathsf{scaleXY}(a,b,s_1),\mathsf{scaleXY}(a,b,s_2))]\!]\\
\mathcal{S}[\![\mathsf{scaleXY}(a,b,\mathsf{difference}(s_1,s_2))]\!] &\cong \mathcal{S}[\![\mathsf{difference}(\mathsf{scaleXY}(a,b,s_1),\mathsf{scaleXY}(a,b,s_2))]\!]\\
\mathcal{S}[\![\mathsf{translate}(p,\mathsf{union}(s_1,s_2))]\!] &\cong \mathcal{S}[\![\mathsf{union}(\mathsf{translate}(p,s_1),\mathsf{translate}(p,s_2))]\!]\\
\mathcal{S}[\![\mathsf{translate}(p,\mathsf{intersect}(s_1,s_2))]\!] &\cong \mathcal{S}[\![\mathsf{intersect}(\mathsf{translate}(p,s_1),\mathsf{translate}(p,s_2))]\!]\\
\mathcal{S}[\![\mathsf{translate}(a,b,\mathsf{difference}(s_1,s_2))]\!] &\cong \mathcal{S}[\![\mathsf{difference}(\mathsf{translate}(p,s_1),\mathsf{translate}(p,s_2))]\!]
\end{aligned}
$$

We can also so the following equations are true

$$
\begin{aligned}
\mathcal{S}[\![\mathsf{scaleXY}(a_0,b_0,\mathsf{scaleXY}(a_1,b_1,s))]\!] &\cong \mathcal{S}[\![\mathsf{scaleXY}(a_0a_1,b_0b_1,s)]\!]\\
\mathcal{S}[\![\mathsf{translate}((x_0,y_0),\mathsf{translate}((x_1,y_1),s))]\!] &\cong \mathcal{S}[\![\mathsf{translate}((x_0+x_1,y_0+y_1),s)]\!]\\
\mathcal{S}[\![\mathsf{scaleXY}(a,b,\mathsf{translate}((x_0,y_0),s))]\!] &\cong \mathcal{S}[\![\mathsf{translate}((ax_0,by_0),\mathsf{scaleXY}(a,b,s))]\!]
\end{aligned}
$$

Finally we have the following set of equations

$$
\begin{aligned}
\mathcal{S}[\![\mathsf{scaleXY}(a,b,\mathsf{ellipse}((x_0,y_0),a_0,b_0))]\!] &\cong \mathcal{S}[\![\mathsf{ellipse}((ax_0,by_0),(aa_0,bb_0))]\!]\\
\mathcal{S}[\![\mathsf{translate}(p_0,\mathsf{ellipse}(p_1,a,b))]\!] &\cong \mathcal{S}[\![\mathsf{ellipse}(p_0+p_1,a,b)]\!]\\
\mathcal{S}[\![\mathsf{scaleXY}(a,b,\mathsf{halfplane}((x_0,y_0),(x_1,y_1)))]\!] &\cong \mathcal{S}[\![\mathsf{halfplane}((ax_0,by_0),(ax_1,by_1))]\!]\\
\mathcal{S}[\![\mathsf{translate}(p_0,\mathsf{halfplane}(p_1,p_2))]\!] &\cong \mathcal{S}[\![\mathsf{halfplane}(p_1+p_0,p_2+p_0)]\!]
\end{aligned}
$$

All the previous equations allow us to simplify shapes so that they are free of any scale or translation operations.

3.2.1 Scaling Distributes Over Union

To see why the equation

$$\mathcal{S}[\![\mathsf{scaleXY}(a,b,\mathsf{union}(s_1,s_2))]\!] \cong \mathcal{S}[\![\mathsf{union}(\mathsf{scaleXY}(a,b,s_1),\mathsf{scaleXY}(a,b,s_2))]\!]$$

holds we can expand both sides to obtain the equation

$$
\begin{aligned}
&\{(x,y) \mid \exists x_s,y_s,\ x=ax_s \ \wedge\ y=by_s \ \wedge\ (x_s,y_s)\in\mathcal{S}[\![\mathsf{union}(s_1,s_2)]\!]\} \cong\\
&\quad\{(x,y) \mid (x,y)\in\mathsf{scaleXY}(a,b,s_1) \ \vee\ (x,y)\in\mathsf{scaleXY}(a,b,s_2)\}
\end{aligned}
$$

We can expand the definitions again to obtain

$$
\begin{aligned}
&\{(x,y) \mid \exists x_s,y_s\ x=ax_s \ \wedge\ y=by_s \ \wedge\\
&\qquad (x_s,y_s)\in\{(x,y)\mid (x,y)\in\mathcal{S}[\![s_1]\!] \ \vee\ (x,y)\in\mathcal{S}[\![s_2]\!]\}\} \cong\\
&\quad\{(x,y) \mid\\
&\qquad (x,y)\in\{(x,y)\mid \exists x_s,y_s\ x=ax_s \ \wedge\ y=by_s \ \wedge\ (x_s,y_s)\in\mathcal{S}[\![s_1]\!]\} \ \vee\\
&\qquad (x,y)\in\{(x,y)\mid \exists x_s,y_s\ x=ax_s \ \wedge\ y=by_s \ \wedge\ (x_s,y_s)\in\mathcal{S}[\![s_2]\!]\}\}
\end{aligned}
$$

Simplifying both sides we have

$$\begin{aligned}&\{(x,y) \mid \exists x_s, y_s\ x = ax_s\ \wedge\ y = by_s\ \wedge\ (x_s,y_s) \in \mathcal{S}[\![s_1]\!]\ \vee\ (x_s,y_s) \in \mathcal{S}[\![s_2]\!]\} \cong\\ &\quad \{(x,y) \mid \exists x_s, y_s\ x = ax_s\ \wedge\ y = by_s\ \wedge\ (x_s,y_s) \in \mathcal{S}[\![s_1]\!]\ \vee\\ &\qquad\qquad \exists x_s, y_s\ x = ax_s\ \wedge\ y = by_s\ \wedge\ (x_s,y_s) \in \mathcal{S}[\![s_2]\!]\}\end{aligned}$$

The r.h.s. can be shown logically equivalent to

$$\begin{aligned}&\{(x,y) \mid \exists x_s, y_s\ x = ax_s\ \wedge\ y = by_s\ \wedge\ (x_s,y_s) \in \mathcal{S}[\![s_1]\!]\ \vee\ (x_s,y_s) \in \mathcal{S}[\![s_2]\!]\} \cong\\ &\quad \{(x,y) \mid \exists x_s, y_s\ x = ax_s\ \wedge\ y = by_s\ \wedge\ (x_s,y_s) \in \mathcal{S}[\![s_1]\!]\ \vee\\ &\qquad\qquad x = ax_s\ \wedge\ y = by_s\ \wedge\ (x_s,y_s) \in \mathcal{S}[\![s_2]\!]\}\end{aligned}$$

The r.h.s. can again be simplified to

$$\begin{aligned}&\{(x,y) \mid \exists x_s, y_s\ x = ax_s\ \wedge\ y = by_s\ \wedge\ (x_s,y_s) \in \mathcal{S}[\![s_1]\!]\ \vee\ (x_s,y_s) \in \mathcal{S}[\![s_2]\!]\} \cong\\ &\quad \{(x,y) \mid \exists x_s, y_s\ x = ax_s\ \wedge\ y = by_s\ \wedge\ (x_s,y_s) \in \mathcal{S}[\![s_1]\!]\ \vee\ (x_s,y_s) \in \mathcal{S}[\![s_2]\!]\}\end{aligned}$$

which is trivially true.

3.2.2 Scaling Distributes Over Intersection

The by similar reasoning the following equation

$$\mathcal{S}[\![\mathsf{scaleXY}(a,b,\mathsf{union}(s_1,s_2))]\!] \cong \mathcal{S}[\![\mathsf{union}(\mathsf{scaleXY}(a,b,s_1),\mathsf{scaleXY}(a,b,s_2))]\!]$$

holds if

$$\begin{aligned}&\{(x,y) \mid \exists x_s, y_s\ x = ax_s\ \wedge\ y = by_s\ \wedge\ (x_s,y_s) \in \mathcal{S}[\![s_1]\!]\ \wedge\ (x_s,y_s) \in \mathcal{S}[\![s_2]\!]\} \cong\\ &\quad \{(x,y) \mid \exists x_s, y_s\ x = ax_s\ \wedge\ y = by_s\ \wedge\ (x_s,y_s) \in \mathcal{S}[\![s_1]\!]\ \wedge\\ &\qquad\qquad \exists x_s, y_s\ x = ax_s\ \wedge\ y = by_s\ \wedge\ (x_s,y_s) \in \mathcal{S}[\![s_2]\!]\}\end{aligned}$$

The r.h.s. is logically equivalent to

$$\begin{aligned}&\{(x,y) \mid \exists x_s, y_s\ x = ax_s\ \wedge\ y = by_s\ \wedge\ (x_s,y_s) \in \mathcal{S}[\![s_1]\!]\ \wedge\ (x_s,y_s) \in \mathcal{S}[\![s_2]\!]\} \cong\\ &\quad \{(x,y) \mid \exists x_s, y_s, x'_s, y'_s\ x = ax_s\ \wedge\ y = by_s\ \wedge\ (x_s,y_s) \in \mathcal{S}[\![s_1]\!]\ \wedge\\ &\qquad\qquad x = ax'_s\ \wedge\ y = by'_s\ \wedge\ (x'_s,y'_s) \in \mathcal{S}[\![s_2]\!]\}\end{aligned}$$

Since $x = ax_s$ and $x = ax'_s$ we must have $x'_s = x_s$ similarly we must have $y_s = y'_s$. From these observations we can simplify the r.h.s again to obtain the trivially true statement

$$\begin{aligned}&\{(x,y) \mid \exists x_s, y_s\ x = ax_s\ \wedge\ y = by_s\ \wedge\ (x_s,y_s) \in \mathcal{S}[\![s_1]\!]\ \wedge\ (x_s,y_s) \in \mathcal{S}[\![s_2]\!]\} \cong\\ &\quad \{(x,y) \mid \exists x_s, y_s\ x = ax_s\ \wedge\ y = by_s\ \wedge\ (x_s,y_s) \in \mathcal{S}[\![s_1]\!]\ \wedge\ (x_s,y_s) \in \mathcal{S}[\![s_2]\!]\}\end{aligned}$$

3.2.3 Scaling Distributes Over Difference

By a similar argument to our last we can show

$$\mathcal{S}[\![\mathsf{scaleXY}(a,b,\mathsf{difference}(s_1,s_2))]\!] \cong \mathcal{S}[\![\mathsf{difference}(\mathsf{scaleXY}(a,b,s_1),\mathsf{scaleXY}(a,b,s_2))]\!]$$

3.2.4 Translation Distributes Over All Set Operations

We can show translation distributes over the set operations in a similar way.

3.2.5 Scale and Translation Compositions

The following equation holds

$$\mathcal{S}[\![\mathsf{scaleXY}(a_0, b_0, \mathsf{scaleXY}(a_1, b_1, s))]\!] \cong \mathcal{S}[\![\mathsf{scaleXY}(a_0 a_1, b_0 b_1, s)]\!]$$

because by expanding our definitions we have

$$\begin{aligned}&\{(x, y) \mid \exists x_s, y_s\ x = a_0 x_s \ \wedge\ y = b_0 y_s \ \wedge\ (x_s, y_s) \in \mathcal{S}[\![\mathsf{scaleXY}(a_1, b_1, s)]\!]\} \cong \\ &\quad \{(x, y) \mid \exists x_s, y_s\ x = a_0 a_1 x_s \ \wedge\ y = b_0 b_1 y_s \ \wedge\ (x_s, y_s) \in \mathcal{S}[\![s]\!]\}\end{aligned}$$

Expanding once more gives us

$$\begin{aligned}&\{(x, y) \mid \exists x_s, y_s\ x = a_0 x_s \ \wedge\ y = b_0 y_s \ \wedge \\ &\quad (x_s, y_s) \in \{(x, y) \mid \exists x_s, y_s\ x = a_1 x_s \ \wedge\ y = b_1 y_s \ \wedge\ (x_s, y_s) \in \mathcal{S}[\![s]\!]\}\} \cong \\ &\{(x, y) \mid \exists x_s, y_s\ x = a_0 a_1 x_s \ \wedge\ y = b_0 b_1 y_s \ \wedge\ (x_s, y_s) \in \mathcal{S}[\![s]\!]\}\end{aligned}$$

We can simplify the l.h.s. resulting in

$$\begin{aligned}&\{(x, y) \mid \exists x_s, y_s, x_s', y_s'\ x = a_0 x_s \ \wedge\ y = b_0 y_s \ \wedge \\ &\quad x_s = a_1 x_s' \ \wedge\ y_s = b_1 y_s' \ \wedge\ (x_s', y_s') \in \mathcal{S}[\![s]\!]\} \cong \\ &\{(x, y) \mid \exists x_s, y_s\ x = a_0 a_1 x_s \ \wedge\ y = b_0 b_1 y_s \ \wedge\ (x_s, y_s) \in \mathcal{S}[\![s]\!]\}\end{aligned}$$

Since we have $x_s = a_1 x_s'$ and $y_s = b_1 y_s'$ we can simplify the l.h.s. to

$$\begin{aligned}&\{(x, y) \mid \exists x_s', y_s'\ x = a_0 a_1 x_s' \ \wedge\ y = b_0 b_1 y_s' \ \wedge\ (x_s', y_s') \in \mathcal{S}[\![s]\!]\} \cong \\ &\quad \{(x, y) \mid \exists x_s, y_s\ x = a_0 a_1 x_s \ \wedge\ y = b_0 b_1 y_s \ \wedge\ (x_s, y_s) \in \mathcal{S}[\![s]\!]\}\end{aligned}$$

which is trivially true.
The equation

$$\mathcal{S}[\![\mathsf{translate}((x_0, y_0), \mathsf{translate}((x_1, y_1), s))]\!] \cong \mathcal{S}[\![\mathsf{translate}((x_0 + x_1, y_0 + y_1), s)]\!]$$

is true via a similar argument.
The equation

$$\mathcal{S}[\![\mathsf{scaleXY}(a, b, \mathsf{translate}((x_0, y_0), s))]\!] \cong \mathcal{S}[\![\mathsf{translate}((a x_0, b y_0), \mathsf{scaleXY}(a, b, s))]\!]$$

is justified since by expanding definitions we obtain the equation

$$\begin{aligned}&\{(x, y) \mid \exists x_s, y_s\ x = a x_s \ \wedge\ y = b y_s \ \wedge\ (x_s, y_s) \in \mathcal{S}[\![\mathsf{translate}((x_0, y_0), s)]\!]\} \cong \\ &\quad \{(x, y) \mid \exists x_s, y_s.\ x = x_s + a x_0 \ \wedge\ y = y_s + b y_0 \ \wedge\ (x_s, y_s) \in \mathsf{scaleXY}(a, b, s)\}\end{aligned}$$

By expanding definitions once more we obtain the equation

$$
\begin{array}{l}
\{(x,y) \mid \exists x_s, y_s\; x = a x_s \;\wedge\; y = b y_s \;\wedge \\
\quad (x_s,y_s) \in \{(x,y) \mid \exists x_s, y_s.\; x = x_s + x_0 \;\wedge\; y = y_s + y_0 \;\wedge\; (x_s,y_s) \in \mathcal{S}[\![s]\!]\}\} \cong \\
\{(x,y) \mid \exists x_s, y_s.\; x = x_s + a x_0 \;\wedge\; y = y_s + b y_0 \;\wedge \\
\quad (x_s,y_s) \in \{(x,y) \mid \exists x_s, y_s\; x = a x_s \;\wedge\; y = b y_s \;\wedge\; (x_s,y_s) \in \mathcal{S}[\![s]\!]\}\}
\end{array}
$$

Simplifying we obtain

$$
\begin{array}{l}
\{(x,y) \mid \exists x_s, y_s, x'_s, y'_s.\; x = a x_s \;\wedge\; y = b y_s \;\wedge\; x_s = x'_s + x_0 \;\wedge\; y_s = y'_s + y_0 \;\wedge \\
\quad (x'_s, y'_s) \in \mathcal{S}[\![s]\!]\} \cong \\
\{(x,y) \mid \exists x_s, y_s, x'_s, y'_s.\; x = x_s + a x_0 \;\wedge\; y = y_s + b y_0 \;\wedge\; x_s = a x'_s \;\wedge\; y_s = b y'_s \;\wedge \\
\quad (x'_s, y'_s) \in \mathcal{S}[\![s]\!]\}
\end{array}
$$

In the l.h.s. we have $x_s = x'_s + x_0$ and $y_s = y'_s + y_0$. In the r.h.s. we have $x_s = a x'_s$ and $y_s = b y'_s$. Using both facts we can simplify the equation to

$$
\begin{array}{c}
\{(x,y) \mid \exists x'_s, y'_s.\; x = a(x'_s + x_0) \;\wedge\; y = b(y'_s + y_0) \;\wedge\; (x'_s, y'_s) \in \mathcal{S}[\![s]\!]\} \cong \\
\{(x,y) \mid \exists x'_s, y'_s.\; x = a x'_s + a x_0 \;\wedge\; y = b y'_s + b y_0 \;\wedge\; (x'_s, y'_s) \in \mathcal{S}[\![s]\!]\}
\end{array}
$$

Collecting like terms in the r.h.s. leads us to the trivially true equation

$$
\begin{array}{c}
\{(x,y) \mid \exists x'_s, y'_s.\; x = a(x'_s + x_0) \;\wedge\; y = b(y'_s + y_0) \;\wedge\; (x'_s, y'_s) \in \mathcal{S}[\![s]\!]\} \cong \\
\{(x,y) \mid \exists x'_s, y'_s.\; x = a(x'_s + x_0) \;\wedge\; y = b(y'_s + y_0) \;\wedge\; (x'_s, y'_s) \in \mathcal{S}[\![s]\!]\}
\end{array}
$$

3.2.6 Scaling an Ellipse

We will show the following equation holds

$$
\mathcal{S}[\![\mathsf{scaleXY}(a, b, \mathsf{ellipse}((x_0, y_0), a_0, b_0))]\!] \cong \mathcal{S}[\![\mathsf{ellipse}((a x_0, b y_0), (a a_0, b b_0))]\!]
$$

First we expand definitions to obtain

$$
\begin{array}{l}
\{(x,y) \mid \exists x_s, y_s\; x = a x_s \;\wedge\; y = b y_s \;\wedge\; (x_s, y_s) \in \mathcal{S}[\![\mathsf{ellipse}((x_0, y_0), a_0, b_0)]\!]\} \cong \\
\quad \{(x,y) \mid (x - a x_0)^2/(a a_0)^2 + (y - b y_0)^2/(b b_0)^2 \leq 1\}
\end{array}
$$

Expanding definitions in the l.h.s. again gives us

$$
\begin{array}{l}
\{(x,y) \mid \exists x_s, y_s\; x = a x_s \;\wedge\; y = b y_s \;\wedge \\
\quad (x_s, y_s) \in \{(x,y) \mid (x - x_0)^2/a_0^2 + (y - y_0)^2/b_0^2 \leq 1\}\} \cong \\
\{(x,y) \mid (x - a x_0)^2/(a a_0)^2 + (y - b y_0)^2/(b b_0)^2 \leq 1\}
\end{array}
$$

Simplifying leaves us with

$$
\begin{array}{l}
\{(x,y) \mid \exists x_s, y_s\; x = a x_s \;\wedge\; y = b y_s \;\wedge\; (x_s - x_0)^2/a_0^2 + (y_s - y_0)^2/b_0^2 \leq 1\} \cong \\
\quad \{(x,y) \mid (x - a x_0)^2/(a a_0)^2 + (y - b y_0)^2/(b b_0)^2 \leq 1\}
\end{array}
$$

On the l.h.s we have $x = ax_s$ so that $x_s = x/a$ likewise for $y = by_s$ we have $y_s = y/b$. Simplifying we have

$$\{(x, y) \mid (x/a - x_0)^2/a_0^2 + (y/b - y_0)^2/b_0^2 \leq 1\} \cong \\ \{(x, y) \mid (x - ax_0)^2/(aa_0)^2 + (y - by_0)^2/(bb_0)^2 \leq 1\}$$

Factoring terms on the l.h.s we have

$$\{(x, y) \mid ((1/a)(x - ax_0))^2/a_0^2 + ((1/b)(y - by_0))^2/b_0^2 \leq 1\} \cong \\ \{(x, y) \mid (x - ax_0)^2/(aa_0)^2 + (y - by_0)^2/(bb_0)^2 \leq 1\}$$

which is the same as

$$\{(x, y) \mid (1/a)^2(x - ax_0)^2(1/a_0^2) + (1/b)^2(y - by_0)^2(1/b_0^2) \leq 1\} \cong \\ \{(x, y) \mid (x - ax_0)^2/(aa_0)^2 + (y - by_0)^2/(bb_0)^2 \leq 1\}$$

Finally we have

$$\{(x, y) \mid (x - ax_0)^2/(aa_0)^2) + (y - by_0)^2/(bb_0)^2 \leq 1\} \cong \\ \{(x, y) \mid (x - ax_0)^2/(aa_0)^2 + (y - by_0)^2/(bb_0)^2 \leq 1\}$$

3.2.7 Translation of an Ellipse

We will show the following equation holds

$$\mathcal{S}[\![\mathsf{translate}((x_0, y_0), \mathsf{ellipse}((x_1, y_1), a, b))]\!] \cong \mathcal{S}[\![\mathsf{ellipse}((x_0 + x_1, y_0 + y_1), a, b)]\!]$$

By expanding definitions we have

$$\{(x, y) \mid \exists x_s, y_s.\ x = x_s + x_0 \ \wedge\ y = y_s + y_0 \ \wedge\ (x_s, y_s) \in \mathcal{S}[\![\mathsf{ellipse}((x_1, y_1), a, b)]\!]\} \cong \\ \{(x, y) \mid (x - (x_0 + x_1))^2/a^2 + (y - (y_0 + y_1))^2/b^2 \leq 1\}$$

Expanding definitions in the l.h.s agian we have

$$\{(x, y) \mid \exists x_s, y_s.\ x = x_s + x_0 \ \wedge\ y = y_s + y_0 \ \wedge \\ (x_s, y_s) \in \{(x, y) \mid (x - x_1)^2/a^2 + (y - y_1)^2/b^2 \leq 1\}\} \cong \\ \{(x, y) \mid (x - (x_0 + x_1))^2/a^2 + (y - (y_0 + y_1))^2/b^2 \leq 1\}$$

Simplifying we have

$$\{(x, y) \mid \exists x_s, y_s.\ x = x_s + x_0 \ \wedge\ y = y_s + y_0 \ \wedge\ (x_s - x_1)^2/a^2 + (y_s - y_1)^2/b^2 \leq 1\} \cong \\ \{(x, y) \mid (x - (x_0 + x_1))^2/a^2 + (y - (y_0 + y_1))^2/b^2 \leq 1\}$$

In the l.h.s we know that $x = x_s + x_0$ so that $x_s = x - x_0$ like wise we know have $y_s = y - y_0$, therefore we can simplify the l.h.s as

$$\{(x, y) \mid ((x - x_0) - x_1)^2/a^2 + ((y - y_0) - y_1)^2/b^2 \leq 1\} \cong \\ \{(x, y) \mid (x - (x_0 + x_1))^2/a^2 + (y - (y_0 + y_1))^2/b^2 \leq 1\}$$

Rearranging the l.h.s. we obtain the trivial equality

$$\{(x, y) \mid (x - (x_0 + x_1))^2/a^2 + (y - (y_0 + y_1))^2/b^2 \leq 1\} \cong \\ \{(x, y) \mid (x - (x_0 + x_1))^2/a^2 + (y - (y_0 + y_1))^2/b^2 \leq 1\}$$

3.2.8 Scaling a Halfplane

3.2.9 Translation of a Halfplane