COS 341 Discrete Mathematics

Generating Functions

Power series

Infinite series of the form $a_0 + a_1 x + a_2 x^2 + \cdots$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

Series converges for x in the interval (-1,1)

Function contains all the information about series

Differentiate k times and substitute x=0,
we get k! times coefficient of x^k

Taylor series of the function $\frac{1}{1-x}$ at $x = 0$

Power series

$(a_0, a_1, a_2, \ldots)$: sequence of real numbers

$|a_n| \leq K^n$

For any number $x \in (-\frac{1}{K}, \frac{1}{K})$, the series

$a(x) = \sum_{i=0}^{\infty} a_i \cdot x^i$ converges

Values of a(x) in arbitrarily small neighborhod of 0 uniquely determine $(a_0, a_1, a_2, \ldots)$

$$a_n = \frac{a^{(n)}(0)}{n!}$$

Generating functions

$(a_0, a_1, a_2, \ldots)$: sequence of real numbers

Generating function of this sequence is

the power series $a(x) = \sum_{i=0}^{\infty} a_i \cdot x^i$

Generating function toolkit: Generalized binomial theorem

$$\binom{r}{k} = \frac{r(r-1)(r-2)\ldots(r-k+1)}{k!}$$

$(1+x)^r$ is the generating function

for the sequence $\left(\binom{r}{0},\binom{r}{1},\binom{r}{2},\binom{r}{3},\ldots\right)$

The power series $\binom{r}{0}+\binom{r}{1}x+\binom{r}{2}x^2+\binom{r}{3}x^3+\ldots$

always converges for all $|x|<1$

Negative binomial coefficients ?

$$\binom{r}{k} = (-1)^k\binom{-r+k-1}{k} = (-1)^k\binom{-r+k-1}{-r-1}$$

$$\frac{1}{(1-x)^n} = \binom{n-1}{n-1}+\binom{n}{n-1}x+\binom{n+1}{n-1}x^2+\cdots+\binom{n+k-1}{n-1}x^k+\ldots$$

$$\frac{1}{1-x} = 1+x+x^2+\cdots$$

Operations on power series

- Addition

  $(a_0+b_0, a_1+b_1, \ldots)$ has generating function $a(x)+b(x)$
- Multiplication by fixed real number

  $(\alpha a_0, \alpha a_1, \ldots)$ has generating function $\alpha a(x)$
- Shifting the sequence to the right

  $(\underbrace{0,\ldots0}_{n\times}, a_0, a_1, \ldots)$ has generating function $x^n a(x)$
- Shifting to the left

  $(a_k, a_{k+1}, \ldots)$ has generating function $\dfrac{a(x)-\sum_{i=0}^{k-1} a_i \cdot x^i}{x^n}$

- Substituting αx for x

  $(a_0, \alpha a_1, \alpha^2 a_2 \ldots)$ has generating function $a(\alpha x)$

  $(1,2,4,8,\ldots)$ has generating function ?
- Substitute x^n for x

  $(a_0, \underbrace{0,\ldots0}_{n-1\times}, a_1, \underbrace{0,\ldots0}_{n-1\times}, a_2 \ldots)$ has generating function $a(x^n)$

  $(1,1,2,2,4,4,8,8,\ldots)$ has generating function ?

  $\dfrac{1}{1-2x^2}+\dfrac{x}{1-2x^2}$

- Differentiation

$(a_1, 2a_2, 3a_3 \ldots)$ has generating function $\frac{d}{dx}a(x)$ (or $a'(x)$)

- Integration

$(0, a_0, \frac{1}{2}a_1, \frac{1}{3}a_2 \ldots)$ has generating function $\int_0^x f(t)dt$

- Multiplication of generating functions

$$\left(\sum_{n=0}^{\infty} a_n \cdot x^n\right)\left(\sum_{n=0}^{\infty} b_n \cdot x^n\right) = \left(\sum_{n=0}^{\infty} c_n \cdot x^n\right)$$

$$c_n = \sum_{k=0}^{n} a_k \cdot b_{n-k}$$

Applying the toolkit

What is the generating function for the sequence $(1^2, 2^2, 3^2, \ldots)$

$$a_k = (k+1)^2$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$\frac{2}{(1-x)^3} = \frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = 1\cdot 2 + 3\cdot 2x + 4\cdot 3x^2 + 5.4x^3 + \cdots$$

$$\frac{2}{(1-x)^3} - \frac{1}{(1-x)^2} = 1\cdot 1 + 2\cdot 2x + 3\cdot 3x^2 + 4\cdot 4x^3 + \cdots$$

An alternate derivation: Generalized Binomial Theorem

$$(1+x)^r = \binom{r}{0} + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \ldots$$

$$\binom{r}{k} = \frac{r(r-1)(r-2)\ldots(r-k+1)}{k!}$$

$$\binom{-n}{k} = \frac{-n(-n-1)(-n-2)\cdots(-n-k+1)}{k!}$$

$$= (-1)^k \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!}$$

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k} = (-1)^k \binom{n+k-1}{n-1}$$

An alternate derivation: Generalized Binomial Theorem

$$(1+x)^r = \binom{r}{0} + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \ldots$$

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k} = (-1)^k \binom{n+k-1}{n-1}$$

$$(1+x)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{n-1} x^k$$

$$(1-x)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{n-1} (-x)^k = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k$$

An alternate derivation: Generalized Binomial Theorem

$$(1-x)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{n-1} (-x)^k = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k$$

$$(1-x)^{-1} = \sum_{k=0}^{\infty} \binom{k}{0} x^k = \sum_{k=0}^{\infty} x^k$$

$$(1-x)^{-2} = \sum_{k=0}^{\infty} \binom{k+1}{1} x^k = \sum_{k=0}^{\infty} (k+1) x^k$$

$$(1-x)^{-3} = \sum_{k=0}^{\infty} \binom{k+2}{2} x^k = \sum_{k=0}^{\infty} \frac{(k+2)(k+1)}{2} x^k$$

An alternate derivation: Generalized Binomial Theorem

$$(1-x)^{-2} = \sum_{k=0}^{\infty} \binom{k+1}{1} x^k = \sum_{k=0}^{\infty} (k+1) x^k$$

$$(1-x)^{-3} = \sum_{k=0}^{\infty} \binom{k+2}{2} x^k = \sum_{k=0}^{\infty} \frac{(k+2)(k+1)}{2} x^k$$

$$2(1-x)^{-3} - (1-x)^{-2} = \sum_{k=0}^{\infty} (k+1)(k+1) x^k$$

More toolkit examples

What is the generating function of the sequence $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$?

$$-\frac{\ln(1-x)}{x} = 1 + \tfrac{1}{2}x + \tfrac{1}{3}x^2 + \tfrac{1}{4}x^3 + \cdots$$

More toolkit examples

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\int_0^x \frac{dt}{1-t} = \int_0^x (1 + t + t^2 + t^3 + t^4 + \cdots) dt$$

$$-\ln(1-x) + \ln(1) = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots$$

$$\frac{-\ln(1-x)}{x} = 1 + \tfrac{1}{2}x + \tfrac{1}{3}x^2 + \tfrac{1}{4}x^3 + \cdots$$

Applications to counting

A box contains 30 red, 40 blue and 50 green balls.
Balls of the same color are indistinguishable.

How many ways are there of selecting a collection of 70 balls from the box ?

coefficient of x^{70} in

$$(1+x+x^2+\cdots+x^{30})$$
$$\times(1+x+x^2+\cdots+x^{40})$$
$$\times(1+x+x^2+\cdots+x^{50})$$

Enter generating functions

coefficient of x^{70} in

$$(1+x+x^2+\cdots+x^{30})(1+x+x^2+\cdots+x^{40})(1+x+x^2+\cdots+x^{50})$$

$$(1+x+x^2+\cdots+x^{30}) = \frac{1-x^{31}}{1-x}$$

Sum of first n terms of a geometric series

Alternately $$\frac{1}{1-x} = 1+x+x^2+\cdots$$

$$\frac{x^{31}}{1-x} = x^{31}+x^{32}+x^{33}+\cdots$$

Enter generating functions

coefficient of x^{70} in

$$(1+x+x^2+\cdots+x^{30})(1+x+x^2+\cdots+x^{40})(1+x+x^2+\cdots+x^{50})$$

coefficient of x^{70} in $\dfrac{(1-x^{31})}{1-x}\dfrac{(1-x^{41})}{1-x}\dfrac{(1-x^{51})}{1-x}$

$$\frac{1}{(1-x)^3}(1-x^{31})(1-x^{41})(1-x^{51})$$

$$= \left(\sum_{k=0}^{\infty}\binom{k+2}{2}x^k\right)(1-x^{31}-x^{41}-x^{51}+\cdots)$$

$$\binom{70+2}{2}-\binom{70-31+2}{2}-\binom{70-41+2}{2}-\binom{70-51+2}{2}=1061$$

More tricks with generating functions

$$a(x) = \sum\nolimits_{i=0}^{\infty} a_i \cdot x^i$$

$$b_n = \sum\nolimits_{i=0}^{n} a_i$$

What is $b(x) = \sum\nolimits_{i=0}^{\infty} b_i \cdot x^i$

$$b(x) = \frac{a(x)}{1-x}$$

$$\sum\nolimits_{i=0}^{\infty} b_i \cdot x^i = \left(a_0 + a_1x + a_2x^2 + \cdots\right)(1+x+x^2+\cdots)$$

More tricks with generating functions

What is $(1^2+2^2+\cdots n^2)$?

$$\frac{2}{(1-x)^3}-\frac{1}{(1-x)^2}=1\cdot 1+2\cdot 2x+3\cdot 3x^2+4\cdot 4x^3+\cdots$$

$$b_n=\sum_{i=0}^{n} a_i \qquad b(x)=\frac{a(x)}{1-x}$$

$$b_0=1^2$$
$$b_1=1^2+2^2$$
$$b_2=1^2+2^2+3^2$$

More tricks with generating functions

What is $(1^2+2^2+\cdots n^2)$?

$$\frac{2}{(1-x)^3}-\frac{1}{(1-x)^2}=1\cdot 1+2\cdot 2x+3\cdot 3x^2+4\cdot 4x^3+\cdots$$

$$\frac{2}{(1-x)^4}-\frac{1}{(1-x)^3}=\sum_{n=0}^{\infty} b_n x^n$$

$$b_n=1^2+2^2+\cdots+(n+1)^2$$

$$b_n=2\binom{3+n}{3}-\binom{2+n}{2}$$

$$b_{n-1}=2\binom{2+n}{3}-\binom{1+n}{2}$$

More tricks with generating functions

What is $(1^2+2^2+\cdots n^2)$?

$$\begin{aligned}
b_{n-1}&=1^2+2^2+\cdots+n^2\\
&=2\binom{2+n}{3}-\binom{1+n}{2}\\
&=\frac{2(n+2)(n+1)n}{6}-\frac{n(n+1)}{2}\\
&=\frac{(2n+1)(n+1)n}{6}
\end{aligned}$$

More tricks with generating functions

What is $\sum_{k=0}^{m}(-1)^k\binom{n}{k}$?

The generating function for the sequence

$a_k=(-1)^k\binom{n}{k}$ is $a(x)=(1-x)^n$

The generating function for the sequence

$c_m=\sum_{k=0}^{m} a_k$ is $\frac{a(x)}{1-x}=(1-x)^{n-1}$

$c_m=$ coefficient of $x^m=(-1)^m\binom{n-1}{m}$