Quick Review

Get paper and pencil out.
Not graded just for review.

Questions

$$\frac{}{\texttt{zero}\ \textbf{int}}\ \text{Z}$$

$$\frac{X\ \textbf{int}}{\texttt{pred(}X\texttt{)}\textbf{int}}\ \text{P} \qquad \frac{X\ \textbf{int}}{\texttt{succ(}X\texttt{)}\textbf{int}}\ \text{S}$$

Write a derivation tree for
`pred(succ(zero))` **int**

$$\frac{}{\texttt{zero}\ \textbf{int}}\ \text{Z}$$

$$\frac{X\ \textbf{int}}{\texttt{pred(}X\texttt{)}\textbf{int}}\ \text{P} \qquad \frac{X\ \textbf{int}}{\texttt{succ(}X\texttt{)}\textbf{int}}\ \text{S}$$

Is the rule below derivable, admissible, or neither?

$$\frac{X\ \textbf{int}}{\texttt{eq(pred(succ(}X\texttt{)),}X\texttt{)}}\ \text{R0}$$

$$\frac{}{\texttt{zero}\ \textbf{int}}\ \text{Z}$$

$$\frac{X\ \textbf{int}}{\texttt{pred(}X\texttt{)}\textbf{int}}\ \text{P} \qquad \frac{X\ \textbf{int}}{\texttt{succ(}X\texttt{)}\textbf{int}}\ \text{S}$$

Is the rule below derivable, admissible, or neither?

$$\frac{\texttt{pred(}X\texttt{)}\textbf{int}}{\texttt{succ(}X\texttt{)}\textbf{int}}\ \text{R1}$$

$$\frac{}{\texttt{zero}\ \textbf{int}}\ \text{Z}$$

$$\frac{X\ \textbf{int}}{\texttt{pred(}X\texttt{)}\textbf{int}}\ \text{P} \qquad \frac{X\ \textbf{int}}{\texttt{succ(}X\texttt{)}\textbf{int}}\ \text{S}$$

Is the rule below derivable, admissible, or neither?

$$\frac{X\ \textbf{int}}{\texttt{succ(pred(}X\texttt{))}\textbf{int}}\ \text{R2}$$

$$\frac{}{\texttt{zero}\ \textbf{int}}\ Z \qquad \frac{X\ \textbf{int}}{\texttt{pred}(X)\ \textbf{int}}\ P \qquad \frac{X\ \textbf{int}}{\texttt{succ}(X)\ \textbf{int}}\ S$$

Is the rule below derivable, admissible, or neither?

$$\frac{\texttt{pred}(X)\ \textbf{int}}{\texttt{pred}(\texttt{succ}(X))\ \textbf{int}}\ R3$$

$$\frac{}{\texttt{zero}\ \textbf{int}}\ Z \qquad \frac{X\ \textbf{int}}{\texttt{pred}(X)\ \textbf{int}}\ P \qquad \frac{X\ \textbf{int}}{\texttt{succ}(X)\ \textbf{int}}\ S$$

Is the rule below derivable, admissible, or neither?

$$\frac{\texttt{succ}(X)\ \textbf{int}}{\texttt{pred}(\texttt{succ}(X))\ \textbf{int}}\ R4$$

$$\frac{}{\texttt{zero}\ \textbf{int}}\ Z \qquad \frac{X\ \textbf{int}}{\texttt{pred}(X)\ \textbf{int}}\ P \qquad \frac{X\ \textbf{int}}{\texttt{succ}(X)\ \textbf{int}}\ S$$

Prove the following

If X **nat** then X **int**

(just sketch out the structure)

$$\frac{}{\texttt{zero}\ \textbf{int}}\ Z \qquad \frac{X\ \textbf{int}}{\texttt{pred}(X)\ \textbf{int}}\ P \qquad \frac{X\ \textbf{int}}{\texttt{succ}(X)\ \textbf{int}}\ S$$

What is the principle of rule induction look like for the rules above?

Answers

$$\frac{}{\texttt{zero}\ \textbf{int}}\ Z \qquad \frac{X\ \textbf{int}}{\texttt{pred}(X)\ \textbf{int}}\ P \qquad \frac{X\ \textbf{int}}{\texttt{succ}(X)\ \textbf{int}}\ S$$

Write a derivation tree for

$$\cfrac{\cfrac{\cfrac{}{\texttt{zero}\ \textbf{int}}\ Z}{\texttt{succ}(\texttt{zero})\ \textbf{int}}\ S}{\texttt{pred}(\texttt{succ}(\texttt{zero}))\ \textbf{int}}\ P$$

$$\frac{}{\texttt{zero}\ \textbf{int}}\ \text{Z}$$

$$\frac{X\ \textbf{int}}{\texttt{pred}(X)\,\textbf{int}}\ \text{P} \qquad \frac{X\ \textbf{int}}{\texttt{succ}(X)\,\textbf{int}}\ \text{S}$$

Is the rule below derivable, admissible, or neither?

$$\frac{X\ \textbf{int}}{\texttt{eq}(\texttt{pred}(\texttt{succ}(X)),X)}\ \text{R0}$$

$$\frac{}{\texttt{zero}\ \textbf{int}}\ \text{Z}$$

$$\frac{X\ \textbf{int}}{\texttt{pred}(X)\,\textbf{int}}\ \text{P} \qquad \frac{X\ \textbf{int}}{\texttt{succ}(X)\,\textbf{int}}\ \text{S}$$

Is the rule below is neither derivable or admissible.

$$\frac{X\ \textbf{int}}{\texttt{eq}(\texttt{pred}(\texttt{succ}(X)),X)}\ \text{R0}$$

$$\frac{}{\texttt{zero}\ \textbf{int}}\ \text{Z}$$

$$\frac{X\ \textbf{int}}{\texttt{pred}(X)\,\textbf{int}}\ \text{P} \qquad \frac{X\ \textbf{int}}{\texttt{succ}(X)\,\textbf{int}}\ \text{S}$$

Is the rule below derivable, admissible, or neither?

$$\frac{\texttt{pred}(X)\,\textbf{int}}{\texttt{succ}(X)\,\textbf{int}}\ \text{R1}$$

$$\frac{}{\texttt{zero}\ \textbf{int}}\ \text{Z}$$

$$\frac{X\ \textbf{int}}{\texttt{pred}(X)\,\textbf{int}}\ \text{P} \qquad \frac{X\ \textbf{int}}{\texttt{succ}(X)\,\textbf{int}}\ \text{S}$$

The rule below is admissible.

$$\frac{\texttt{pred}(X)\,\textbf{int}}{\texttt{succ}(X)\,\textbf{int}}\ \text{R1}$$

$$\frac{}{\texttt{zero}\ \textbf{int}}\ \text{Z}$$

$$\frac{X\ \textbf{int}}{\texttt{pred}(X)\,\textbf{int}}\ \text{P} \qquad \frac{X\ \textbf{int}}{\texttt{succ}(X)\,\textbf{int}}\ \text{S}$$

Is the rule below derivable, admissible, or neither?

$$\frac{X\ \textbf{int}}{\texttt{succ}(\texttt{pred}(X))\,\textbf{int}}\ \text{R2}$$

$$\frac{}{\texttt{zero}\ \textbf{int}}\ \text{Z}$$

$$\frac{X\ \textbf{int}}{\texttt{pred}(X)\,\textbf{int}}\ \text{P} \qquad \frac{X\ \textbf{int}}{\texttt{succ}(X)\,\textbf{int}}\ \text{S}$$

The rule below is derivable.

$$\frac{X\ \textbf{int}}{\texttt{succ}(\texttt{pred}(X))\,\textbf{int}}\ \text{R2}$$

$$\frac{}{\texttt{zero}\ \textbf{int}}\ \text{Z} \qquad \frac{X\ \textbf{int}}{\texttt{pred}(X)\ \textbf{int}}\ \text{P} \qquad \frac{X\ \textbf{int}}{\texttt{succ}(X)\ \textbf{int}}\ \text{S}$$

Is the rule below derivable, admissible, or neither?

$$\frac{\texttt{pred}(X)\ \textbf{int}}{\texttt{pred}(\texttt{succ}(X))\ \textbf{int}}\ \text{R3}$$

$$\frac{}{\texttt{zero}\ \textbf{int}}\ \text{Z} \qquad \frac{X\ \textbf{int}}{\texttt{pred}(X)\ \textbf{int}}\ \text{P} \qquad \frac{X\ \textbf{int}}{\texttt{succ}(X)\ \textbf{int}}\ \text{S}$$

The rule below is admissible.

$$\frac{\texttt{pred}(X)\ \textbf{int}}{\texttt{pred}(\texttt{succ}(X))\ \textbf{int}}\ \text{R3}$$

$$\frac{}{\texttt{zero}\ \textbf{int}}\ \text{Z} \qquad \frac{X\ \textbf{int}}{\texttt{pred}(X)\ \textbf{int}}\ \text{P} \qquad \frac{X\ \textbf{int}}{\texttt{succ}(X)\ \textbf{int}}\ \text{S}$$

Is the rule below derivable, admissible, or neither?

$$\frac{\texttt{succ}(X)\ \textbf{int}}{\texttt{pred}(\texttt{succ}(X))\ \textbf{int}}\ \text{R4}$$

$$\frac{}{\texttt{zero}\ \textbf{int}}\ \text{Z} \qquad \frac{X\ \textbf{int}}{\texttt{pred}(X)\ \textbf{int}}\ \text{P} \qquad \frac{X\ \textbf{int}}{\texttt{succ}(X)\ \textbf{int}}\ \text{S}$$

The rule below is derivable.

$$\frac{\texttt{succ}(X)\ \textbf{int}}{\texttt{pred}(\texttt{succ}(X))\ \textbf{int}}\ \text{R4}$$

$$\frac{}{\texttt{zero}\ \textbf{int}}\ \text{Z} \qquad \frac{X\ \textbf{int}}{\texttt{pred}(X)\ \textbf{int}}\ \text{P} \qquad \frac{X\ \textbf{int}}{\texttt{succ}(X)\ \textbf{int}}\ \text{S}$$

Prove the following

If X **nat** then X **int**

(just sketch out the structure, i.e.)

Proof Sketch

By induction on X nat

IH(x) = If x **nat** then x **int**

Subgoal1: IH(`zero`)

Subgoal2: If IH(X') then IH(`succ(`X'`)`)

$$\frac{}{\texttt{zero}\ \textbf{int}}\ z$$

$$\frac{X\ \textbf{int}}{\texttt{pred}(X)\ \textbf{int}}\ P \qquad \frac{X\ \textbf{int}}{\texttt{succ}(X)\ \textbf{int}}\ S$$

What is the principle of rule induction look like for the rules above?

Rule Induction Principle

If X **int**,
P(zero),
if P(X') then P(succ(X')), and
if P(Y') then P(pred(Y'))
then P(X)

Did You Ace The Quiz?

- If so great!
- If not go through the notes and the slides from lecture 1
- Still stuck talk to me or the TA
- Did the entire class ace the quiz?
  - Probably not you are not the only one who is is confused!

Inductively Defined Functions and Standard ML

COS 441
Princeton University
Fall 2004

Assignment 1

- Handout today due back next Wednesday
- Requires ML programming an a few simple proofs
- Make sure you're all set up to use your CS account and program in ML
- Details and updates available through the course web

Relations Review

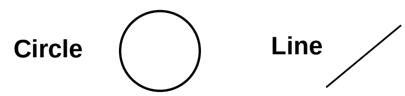
- *A relation* is set of *tuples*
  - **Odd** = {1, 3, 5, … }
  - **Line** = { (0.0, 0.0), (1.5,1.5), (x, x) , …}
  - **Circle** = { (x, y) | $x^2 + y^2 = 1.0$ }
- **Odd** is a *predicate* on natural numbers
- **Line**, **Circle**, and **Sphere** are relations on real numbers
- **Line** is a *function*

Functions and Their Graphs

- The *graph* of a function **f**(x) is the unique relation {(x,y) | **f**(x) = y }
- We can uniquely specify a function by defining its graph as a relation
- Not all relations specify valid functions!

Some "graphs" of Relations

- Below are some plotted graphs of the relations **Circle** and **Line**



- For a relation to be a valid graph of a function each unique input has a unique output

Defining the Function **add**

- We want to define a function **add**(m,n)
- To do this first specify a relation that defines its graph **A**(m,n,p) inductively
- Next show that our for any unique pair of m and n there is a **unique** p such that **A**(m,n,p)

Defining the Graph of **add**

$$\frac{X\ \textbf{nat}}{\mathbf{A}(X,\texttt{zero},X)}\ \text{A-Z} \qquad \frac{\mathbf{A}(X,Y,Z)}{\mathbf{A}(X,\texttt{succ}(Y),\texttt{succ}(Z))}\ \text{A-S}$$

Avoiding Clutter

Alternative definition that is equivalent to our previous one but its more cluttered since we have redundant premises

$$\frac{X\ \textbf{nat}}{\mathbf{A}(X,\texttt{zero},X)}\ \text{A-Z}$$

$$\frac{X\ \textbf{nat} \quad Y\ \textbf{nat} \quad Z\ \textbf{nat} \quad \mathbf{A}(X,Y,Z)}{\mathbf{A}(X,\texttt{succ}(Y),\texttt{succ}(Z))}\ \text{A-S}$$

Why are the all those extra premises not needed?

Defining the Graph of **add**

$$\frac{X\ \textbf{nat}}{\mathbf{A}(X,\texttt{zero},X)}\ \text{A-Z} \qquad \frac{\mathbf{A}(X,Y,Z)}{\mathbf{A}(X,\texttt{succ}(Y),\texttt{succ}(Z))}\ \text{A-S}$$

The definition above immediately entails the following rules

$$\frac{\mathbf{A}(X,Y,Z)}{X\ \textbf{nat}}\ \text{A-X-nat} \qquad \frac{\mathbf{A}(X,Y,Z)}{Y\ \textbf{nat}}\ \text{A-Y-nat} \qquad \frac{\mathbf{A}(X,Y,Z)}{Z\ \textbf{nat}}\ \text{A-Z-nat}$$

Why?

Defining the Graph of **add**

$$\frac{X \ \textbf{nat}}{\textbf{A}(X,\texttt{zero},X)}\ \text{A-Z} \qquad \frac{\textbf{A}(X,Y,Z)}{\textbf{A}(X,\texttt{succ}(Y),\texttt{succ}(Z))}\ \text{A-S}$$

The definition above immediately entails the following rules

$$\frac{\textbf{A}(X,Y,Z)}{X\ \textbf{nat}}\ \text{A-X-nat} \qquad \frac{\textbf{A}(X,Y,Z)}{Y\ \textbf{nat}}\ \text{A-Y-nat} \qquad \frac{\textbf{A}(X,Y,Z)}{Z\ \textbf{nat}}\ \text{A-Z-nat}$$

They can be shown to be admissible with the principle of rule induction for derivations of A and the rules Z and S

Proving **A** is a Function Graph

If **A**(X,Y,Z), X unique, and Y unique then Z is unique.

Proof: By ??

Proving **A** is a Function Graph

If **A**(X,Y,Z), X unique, and Y unique then Z is unique.

Proof: By rule induction for **A**(X,Y,Z)

Proving **A** is a Function Graph

If **A**(X,Y,Z), X unique, and Y unique then Z is unique.

Proof: By rule induction for **A**(X,Y,Z)

If **A**(X,Y,Z),

If X' nat then P(X',`zero`,X'), and

If P(X',Y',Z') then P(X',`succ`(Y'),`succ`(Z'))

then P(X,Y,Z).

Proving **A** is a Function Graph

If **A**(X,Y,Z),

case A-Z: If X' nat then

IH(X',`zero`,X'),

case A-S: If IH(X',Y',Z') then

IH(X',`succ`(Y'),`succ`(Z'))

then IH(X,Y,Z).

Proving **A** is a Function Graph

If **A**(X,Y,Z),

case A-Z: If X' nat then

IH(X',`zero`,X'),

case A-S: If IH(X',Y',Z') then

IH(X',`succ`(Y'),`succ`(Z'))

then IH(X,Y,Z).

IH(x,y,z) = ??

Proving **A** is a Function Graph

If **A**(X,Y,Z),
  case A-Z: If X' nat then
    IH(X',`zero`,X'),
  case A-S: If IH(X',Y',Z') then
    IH(X',`succ(`Y'`)`,`succ(`Z'`))`
then IH(X,Y,Z).
IH(x,y,z) = If **A**(x,y,z), x unique, and y unique then
  z is unique.

Proving **A** is a Function Graph

case A-Z: If X' nat then IH(X',`zero`,X')
IH(x,y,z) = If **A**(x,y,z), x unique, and y unique then
  z is unique.

Proving **A** is a Function Graph

case A-Z: If X' nat then if **A**(X',`zero`,X'), X' unique,
  and `zero` unique
  then X' is unique.

Proving **A** is a Function Graph

case A-Z: … then if **A**(X',`zero`,X'), X' unique,
  and `zero` unique
  then X' is unique.

1. X' nat — by assumption

Proving **A** is a Function Graph

case A-Z: … then X' is unique.

1. X' nat — by assumption
2. **A**(X',`zero`,X'), X' unique, and `zero` unique — by assumption

Proving **A** is a Function Graph

case A-Z:

1. X' nat — by assumption
2. **A**(X',`zero`,X'), X' unique, and `zero` unique — by assumption
3. X' unique — by (2)

Proving **A** is a Function Graph

case A-S: If IH(X',Y',Z') then
IH(X',succ(Y'),succ(Z'))

IH(x,y,z) = If **A**(x,y,z), x unique, and y unique then
z is unique.

Proving **A** is a Function Graph

case A-S: … then
if **A**(X',succ(Y'),succ(Z')), X' unique,
and succ(Y') unique
then succ(Z') is unique

1. IH(X',Y',Z') by assumption

IH(x,y,z) = If **A**(x,y,z), x unique, and y unique then
z is unique.

Proving **A** is a Function Graph

case A-S: … then succ(Z') is unique

1. IH(X',Y',Z') by assumption
2. **A**(X',succ(Y'),succ(Z')), X' unique, and succ(Y') unique by assumption

IH(x,y,z) = If **A**(x,y,z), x unique, and y unique then
z is unique.

Proving **A** is a Function Graph

case A-S: … then succ(Z') is unique

1. IH(X',Y',Z') by assumption
2. **A**(X',succ(Y'),succ(Z')), X' unique, and succ(Y') unique by assumption
3. **A**(X',Y',Z') by ??

IH(x,y,z) = If **A**(x,y,z), x unique, and y unique then
z is unique.

Proving **A** is a Function Graph

case A-S: … then succ(Z') is unique

1. IH(X',Y',Z') by assumption
2. **A**(X',succ(Y'),succ(Z')), X' unique, and succ(Y') unique by assumption
3. **A**(X',Y',Z') by (2) and **invert-A-S**

IH(x,y,z) = If **A**(x,y,z), x unique, and y unique then
z is unique.

Proving **A** is a Function Graph

case A-S: … then succ(Z') is unique

1. IH(X',Y',Z') by assumption
2. **A**(X',succ(Y'),succ(Z')), X' unique, and succ(Y') unique by assumption
3. **A**(X',Y',Z') by (2) and **invert-A-S**
4. Y' unique

IH(x,y,z) = If **A**(x,y,z), x unique, and y unique then
z is unique.

Proving **A** is a Function Graph

case A-S: … then succ(Z') is unique

1. IH(X',Y',Z') by assumption
2. **A**(X',succ(Y'),succ(Z')), X' unique, and succ(Y') unique by assumption
3. **A**(X',Y',Z') by (2) and **invert-A-S**
4. Y' unique by ??

IH(x,y,z) = If **A**(x,y,z), x unique, and y unique then z is unique.

Proving **A** is a Function Graph

case A-S: … then succ(Z') is unique

1. IH(X',Y',Z') by assumption
2. **A**(X',succ(Y'),succ(Z')), X' unique, and succ(Y') unique by assumption
3. **A**(X',Y',Z') by (2) and **invert-A-S**
4. Y' unique by ??
5. Z' unique by ??

IH(x,y,z) = If **A**(x,y,z), x unique, and y unique then z is unique.

Proving **A** is a Function Graph

case A-S: … then succ(Z') is unique

1. IH(X',Y',Z') by assumption
2. **A**(X',succ(Y'),succ(Z')), X' unique, and succ(Y') unique by assumption
3. **A**(X',Y',Z') by (2) and **invert-A-S**
4. Y' unique by ??
5. Z' unique by IH with (3,2,4)

IH(x,y,z) = If **A**(x,y,z), x unique, and y unique then z is unique.

Proving **A** is a Function Graph

case A-S:

1. IH(X',Y',Z') by assumption
2. **A**(X',succ(Y'),succ(Z')), X' unique, and succ(Y') unique by assumption
3. **A**(X',Y',Z') by (2) and **invert-A-S**
4. Y' unique by ??
5. Z' unique by IH with (3,2,4)
6. succ(Z') unique by ??

IH(x,y,z) = If **A**(x,y,z), x unique, and y unique then z is unique.

Some Missing Pieces

$$\frac{\mathbf{A}(X,\mathtt{succ}(Y),\mathtt{succ}(Z))}{\mathbf{A}(X,Y,Z)}\ \textbf{invert-A-S}$$

- We need to assume the following
  - zero is unique
  - If X **nat** and X unique then succ(X) is unique
  - If X **nat** and succ(X) unique then X is unique
- It is okay to assume "obvious" things just be explicit about what you assume in proofs

A Function as Recursive Equations

- We often use different notations to defined the graph of a functions

add(M,zero) ≅ M
add(M,succ(N)) ≅ succ(**add**(M,N))

- The equations define a relation implicitly that relation must still be shown to be the graph of a valid function

Example: Fibonacci Function

$$\begin{aligned} \mathbf{fib}(\mathtt{zero}) &\cong \mathtt{succ(zero)} \\ \mathbf{fib}(\mathtt{succ(zero)}) &\cong \mathtt{succ(zero)} \\ \mathbf{fib}(\mathtt{succ(succ(}N\mathtt{))}) &\cong \mathbf{add}(\mathbf{fib}(\mathtt{succ(}N\mathtt{)}), \\ &\qquad \mathbf{fib}(N)) \end{aligned}$$

What are the rules for the relation being implicitly defined by the equations above?

Example: Fibonacci Function

$$\frac{}{\mathbf{F}(\mathtt{zero},\mathtt{succ(zero)})}\ \text{F-Z}$$

$$\begin{aligned} \mathbf{fib}(\mathtt{succ(zero)}) &\cong \mathtt{succ(zero)} \\ \mathbf{fib}(\mathtt{succ(succ(}N\mathtt{))}) &\cong \mathbf{add}(\mathbf{fib}(\mathtt{succ(}N\mathtt{)}), \\ &\qquad \mathbf{fib}(N)) \end{aligned}$$

Example: Fibonacci Function

$$\frac{}{\mathbf{F}(\mathtt{zero},\mathtt{succ(zero)})}\ \text{F-Z}$$

$$\frac{}{\mathbf{F}(\mathtt{succ(zero)},\mathtt{succ(zero)})}\ \text{F-S-Z}$$

$$\begin{aligned} \mathbf{fib}(\mathtt{succ(succ(}N\mathtt{))}) &\cong \mathbf{add}(\mathbf{fib}(\mathtt{succ(}N\mathtt{)}), \\ &\qquad \mathbf{fib}(N)) \end{aligned}$$

Example: Fibonacci Function

$$\frac{}{\mathbf{F}(\mathtt{zero},\mathtt{succ(zero)})}\ \text{F-Z}$$

$$\frac{}{\mathbf{F}(\mathtt{succ(zero)},\mathtt{succ(zero)})}\ \text{F-S-Z}$$

$$\frac{\mathbf{F}(\mathtt{succ(}N\mathtt{)},X) \quad \mathbf{F}(N,Y) \quad \mathbf{A}(X,Y,Z)}{\mathbf{F}(\mathtt{succ(succ(}N\mathtt{))},Z)}\ \text{F-S-S-N}$$

Example: Fibonacci Function

$$\frac{}{\mathbf{F}(\mathtt{zero},\mathtt{succ(zero)})}\ \text{F-Z}$$

$$\frac{}{\mathbf{F}(\mathtt{succ(zero)},\mathtt{succ(zero)})}\ \text{F-S-Z}$$

$$\frac{\mathbf{F}(\mathtt{succ(}N\mathtt{)},X) \quad \mathbf{F}(N,Y) \quad \mathbf{A}(X,Y,Z)}{\mathbf{F}(\mathtt{succ(succ(}N\mathtt{))},Z)}\ \text{F-S-S-N}$$

Does the relation **F** define the graph of a function?

Why?

Summary of Definitions

$$\frac{}{\mathtt{zero}\ \mathbf{nat}}\ \text{Z} \qquad \frac{X\ \mathbf{nat}}{\mathtt{succ(}X\mathtt{)}\ \mathbf{nat}}\ \text{S}$$

$$\frac{X\ \mathbf{nat}}{\mathbf{A}(X,\mathtt{zero},X)}\ \text{A-Z} \qquad \frac{\mathbf{A}(X,Y,Z)}{\mathbf{A}(X,\mathtt{succ(}Y\mathtt{)},\mathtt{succ(}Z\mathtt{)})}\ \text{A-S}$$

$$\frac{}{\mathbf{F}(\mathtt{zero},\mathtt{succ(zero)})}\ \text{F-Z}$$

$$\frac{}{\mathbf{F}(\mathtt{succ(zero)},\mathtt{succ(zero)})}\ \text{F-S-Z}$$

$$\frac{\mathbf{F}(\mathtt{succ(}N\mathtt{)},X) \quad \mathbf{F}(N,Y) \quad \mathbf{A}(X,Y,Z)}{\mathbf{F}(\mathtt{succ(succ(}N\mathtt{))},Z)}\ \text{F-S-S-N}$$

Some Derivable Judgments

F(succ(succ(zero)),
succ(succ(zero)))
F(succ(succ(succ(zero))),
succ(succ(succ(zero))))
F(succ(succ(succ(succ(zero)))),
succ(succ(succ(succ(succ(zero))
)))) ….

From Relations to SML

- Deriving the judgments by hand is tedious!
- We can use SML as a calculator of sorts to directly express the function we defined as an SML function
- To do this first we have to separate our functions from our data

Separating Functions From Data

- The **nat** predicate defines an *abstract syntax tree*
  - More about this next lecture
- We can express the remaining relations as recursive equations that define a function
  - We need to verify that the equations do define well defined functions
  - But we've done that already for these two relations in this lecture!

Separating Functions From Data

$$\frac{}{\texttt{zero}\ \textbf{nat}}\ \text{Z} \qquad \frac{X\ \textbf{nat}}{\texttt{succ}(X)\ \textbf{nat}}\ \text{S}$$

$$\frac{X\ \textbf{nat}}{\textbf{A}(X,\texttt{zero},X)}\ \text{A-Z} \qquad \frac{\textbf{A}(X,Y,Z)}{\textbf{A}(X,\texttt{succ}(Y),\texttt{succ}(Z))}\ \text{A-S}$$

$$\frac{}{\textbf{F}(\texttt{zero},\texttt{succ}(\texttt{zero}))}\ \text{F-Z}$$

$$\frac{}{\textbf{F}(\texttt{succ}(\texttt{zero}),\texttt{succ}(\texttt{zero}))}\ \text{F-S-Z}$$

$$\frac{\textbf{F}(\texttt{succ}(N),X) \quad \textbf{F}(N,Y) \quad \textbf{A}(X,Y,Z)}{\textbf{F}(\texttt{succ}(\texttt{succ}(N)),Z)}\ \text{F-S-S-N}$$

Separating Functions From Data

nat *n* ::= zero | succ(*n*)

$$\frac{X\ \textbf{nat}}{\textbf{A}(X,\texttt{zero},X)}\ \text{A-Z} \qquad \frac{\textbf{A}(X,Y,Z)}{\textbf{A}(X,\texttt{succ}(Y),\texttt{succ}(Z))}\ \text{A-S}$$

$$\frac{}{\textbf{F}(\texttt{zero},\texttt{succ}(\texttt{zero}))}\ \text{F-Z}$$

$$\frac{}{\textbf{F}(\texttt{succ}(\texttt{zero}),\texttt{succ}(\texttt{zero}))}\ \text{F-S-Z}$$

$$\frac{\textbf{F}(\texttt{succ}(N),X) \quad \textbf{F}(N,Y) \quad \textbf{A}(X,Y,Z)}{\textbf{F}(\texttt{succ}(\texttt{succ}(N)),Z)}\ \text{F-S-S-N}$$

Separating Functions From Data

nat *n* ::= zero | succ(*n*)

$$\textbf{add}(M,\texttt{zero}) \cong M$$
$$\textbf{add}(M,\texttt{succ}(N)) \cong \texttt{succ}(\textbf{add}(M,N))$$

$$\frac{}{\textbf{F}(\texttt{zero},\texttt{succ}(\texttt{zero}))}\ \text{F-Z}$$

$$\frac{}{\textbf{F}(\texttt{succ}(\texttt{zero}),\texttt{succ}(\texttt{zero}))}\ \text{F-S-Z}$$

$$\frac{\textbf{F}(\texttt{succ}(N),X) \quad \textbf{F}(N,Y) \quad \textbf{A}(X,Y,Z)}{\textbf{F}(\texttt{succ}(\texttt{succ}(N)),Z)}\ \text{F-S-S-N}$$

Separating Functions From Data

nat *n* ::= `zero` | `succ(`*n*`)`

add(M,`zero`) ≅ M
add(M,`succ(`N`)`) ≅ `succ(`**add**(M,N)`)`

fib(`zero`) ≅ `succ(zero)`
fib(`succ(zero)`) ≅ `succ(zero)`
fib(`succ(succ(`N`))`) ≅ **add**(**fib**(`succ(`N`)`), **fib**(N))

From Relations to SML (cont.)

- We can convert the abstract syntax tree into and ML **`datatype`** declarations
- The recursive equations we can write down as ML functions
  - What if our recursive equations didn't actually define a function but we translated it naviely anyway?

From Relations to SML (cont.)

nat *n* ::= `zero` | `succ(`*n*`)`

add(M,`zero`) ≅ M
add(M,`succ(`N`)`) ≅ `succ(`**add**(M,N)`)`

fib(`zero`) ≅ `succ(zero)`
fib(`succ(zero)`) ≅ `succ(zero)`
fib(`succ(succ(`N`))`) ≅ **add**(**fib**(`succ(`N`)`), **fib**(N))

From Relations to SML (cont.)

```
datatype nat = zero | succ of nat
```

add(M,`zero`) ≅ M
add(M,`succ(`N`)`) ≅ `succ(`**add**(M,N)`)`

fib(`zero`) ≅ `succ(zero)`
fib(`succ(zero)`) ≅ `succ(zero)`
fib(`succ(succ(`N`))`) ≅ **add**(**fib**(`succ(`N`)`), **fib**(N))

From Relations to SML (cont.)

```
datatype nat = zero | succ of nat

fun add(m,zero)          = m
  | add(m,succ(n))       = succ(add(m,n))
```

fib(`zero`) ≅ `succ(zero)`
fib(`succ(zero)`) ≅ `succ(zero)`
fib(`succ(succ(`N`))`) ≅ **add**(**fib**(`succ(`N`)`), **fib**(N))

From Relations to SML (cont.)

```
datatype nat = zero | succ of nat

fun add(m,zero)          = m
  | add(m,succ(n))       = succ(add(m,n))

fun fib(zero)            = succ(zero)
  | fib(succ(zero))      = succ(zero)
  | fib(succ(succ(n)))   = add(fib(succ(n)),
                              fib(n))
```

Lessons Learned

- We can define functions by inductively specifying a relation that defines it graph
- Recursive equations can be used to specify relations that define functions
  - We must verify that the function is well defined
- Many recursive equations can be turned directly into SML code
  - The reason we use SML in this course

Next Lecture

- Lexical analysis and parsing along with other things you will not learn about in detail from this this course
  - We'll talk about them to understand why they are "uninteresting"
- Abstract Syntax