COS 341 Discrete Mathematics

Exponential Generating Functions

Ordinary Generating Functions

$(a_0, a_1, a_2, \ldots)$: sequence of real numbers

Ordinary Generating Function of this sequence is the power series $a(x) = \sum_{i=0}^{\infty} a_i \cdot x^i$

Exponential Generating Functions

$(a_0, a_1, a_2, \ldots)$: sequence of real numbers

Exponential Generating function of this sequence is the power series

$$a(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot x^i$$

Exponential generating function examples

What is the generating function for the sequence $(1,1,1,1,\ldots)$?

$$\sum_{i=0}^{\infty} \frac{1}{i!} x^i = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$$

What is the generating function for the sequence $(1,2,4,8,\ldots)$?

$$\sum_{i=0}^{\infty} \frac{2^i}{i!} x^i = 1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots = e^{2x}$$

Operations on exponential generating functions

- Addition

  $(a_0+b_0, a_1+b_1, \ldots)$ has generating function $a(x)+b(x)$

- Multiplication by fixed real number

  $(\alpha a_0, \alpha a_1, \ldots)$ has generating function $\alpha a(x)$

- Shifting the sequence to the right

  ~~$(\underbrace{0,\ldots 0}_{n\times}, a_0, a_1, \ldots)$ has generating function $x^n a(x)$~~

- Shifting to the left

  ~~$(a_k, a_{k+1}, \ldots)$ has generating function $\dfrac{a(x)-\sum_{i=0}^{k-1} a_i \cdot x^i}{x^n}$~~

- Substituting αx for x

  $(a_0, \alpha a_1, \alpha^2 a_2 \ldots)$ has generating function $a(\alpha x)$

- Substitute x^n for x

  ~~$(a_0, \underbrace{0,\ldots 0}_{n-1\times}, a_1, \underbrace{0,\ldots 0}_{n-1\times}, a_2 \ldots)$ has generating function $a(x^n)$~~

- Differentiation

  ~~$(a_1, 2a_2, 3a_3 \ldots)$ has generating function $\frac{d}{dx}a(x)$ (or $a'(x)$)~~

- Integration

  ~~$(0, a_0, \frac{1}{2}a_1, \frac{1}{3}a_2 \ldots)$ has generating function $\int_0^x f(t)dt$~~

- Multiplication of generating functions

  ~~$\left(\sum_{n=0}^{\infty} a_n \cdot x^n\right)\left(\sum_{n=0}^{\infty} b_n \cdot x^n\right) = \left(\sum_{n=0}^{\infty} c_n \cdot x^n\right)$~~

  ~~$c_n = \sum_{k=0}^{n} a_k \cdot b_{n-k}$~~

Differentiation

$a(x)$ is the exponential generating function for $(a_0, a_1, a_2, \ldots)$

$$a(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot x^i$$

$$\frac{d}{dx}a(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot i \cdot x^{i-1} = \sum_{i=1}^{\infty} \frac{a_i}{(i-1)!} \cdot x^{i-1}$$

$\frac{d}{dx}a(x)$ is the exponential generating function for $(a_1, a_2, a_3, \ldots)$

Differentiation is equivalent to shifting the sequence to the left

Integration

$a(x)$ is the exponential generating function for $(a_0, a_1, a_2, \ldots)$

$$a(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot x^i$$

$$\int_0^x a(t)dt = \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot \int_0^x t^i dt = \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot \frac{x^{i+1}}{(i+1)} = \sum_{i=0}^{\infty} \frac{a_i}{(i+1)!} \cdot x^{i+1}$$

$\int_0^x a(t)dt$ is the exponential generating function for $(0, a_0, a_1, a_2, \ldots)$

Integration is equivalent to shifting the sequence to the right

Multiplication

Ordinary $\left(\sum_{n=0}^{\infty} a_n \cdot x^n\right)\left(\sum_{n=0}^{\infty} b_n \cdot x^n\right) = \left(\sum_{n=0}^{\infty} c_n \cdot x^n\right)$

$$c_n = \sum_{k=0}^{n} a_k \cdot b_{n-k}$$

Exponential $\left(\sum_{n=0}^{\infty} \frac{a_n}{n!} \cdot x^n\right)\left(\sum_{n=0}^{\infty} \frac{b_n}{n!} \cdot x^n\right) = \left(\sum_{n=0}^{\infty} \frac{c_n}{n!} \cdot x^n\right)$

$$\frac{c_n}{n!} = \sum_{k=0}^{n} \frac{a_k}{k!} \cdot \frac{b_{n-k}}{(n-k)!}$$

$$c_n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a_k \cdot b_{n-k} = \sum_{k=0}^{n} \binom{n}{k} a_k \cdot b_{n-k}$$

Implications of product rule

$$C(x) = A(x)B(x)$$

Ordinary $\boxed{c_n = \sum_{k=0}^{n} a_k \cdot b_{n-k}}$

Useful for counting with indistinguishable objects

Exponential $\boxed{c_n = \sum_{k=0}^{n} \binom{n}{k} a_k \cdot b_{n-k}}$

Useful for counting with ordered objects

Interpretation of Multiplication: Product Rule

Given arrangements of type A and type B, define arrangements of type C for n labeled objects as follows:

Divide the group of n labeled objects into two groups, the First group and the Second group;
arrange the First group by an arrangement of type A
and the Second group by an arrangement of type B.

a_n : number of arrangements of type A for n objects
b_n : number of arrangements of type B for n objects
c_n : number of arrangements of type C for n objects

Interpretation of Multiplication: Product Rule

a_n : number of arrangements of type A for n people

b_n : number of arrangements of type B for n people

c_n : number of arrangements of type C for n people

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k \cdot b_{n-k}$$

a_n : exponential generating function $A(x)$

b_n : exponential generating function $B(x)$

c_n : exponential generating function $C(x)$

$$C(x) = A(x)B(x)$$

$A(x)$: exponential generating function for arrangements of type A

$a_0 = 0$: no empty group allowed

Define arrangements of type D for n labeled objects as follows:

Divide the group of n labeled objects into k groups,
the First group, Second group,…, kth group $(k = 0,1,2,\ldots)$
arrange each group by an arrangement of type A .

$D(x)$: exponential generating function for arrangements of type D

$D_k(x)$: exponential generating function for arrangements of type D with exactly k groups

$$D_k(x) = A(x)^k$$

$$D(x) = \sum_{k=0}^{\infty} D_k(x) = \sum_{k=0}^{\infty} A(x)^k = \frac{1}{1-A(x)}$$

$A(x)$: exponential generating function for arrangements of type A

$a_0 = 0$: no empty group allowed

Define arrangements of type E for n labeled objects as follows:

Divide the group of n labeled objects into k groups,
and arrange each group by an arrangement of type A
(the groups are not numbered) .

$E(x)$: exponential generating function for arrangements of type E

$E_k(x)$: exponential generating function for arrangements of type E with exactly k groups

$$E_k(x) = \frac{A(x)^k}{k!}$$

$$E(x) = \sum_{k=0}^{\infty} E_k(x) = \sum_{k=0}^{\infty} \frac{A(x)^k}{k!} = e^{A(x)}$$

Example

How many sequences of n letters can be formed from A, B, and C such that the number of A's is odd and the number of B's is odd ?

$$\text{EGF for A's} = \sum_{n \text{ odd}} \frac{x^n}{n!} = \frac{e^x - e^{-x}}{2}$$

$$\text{EGF for B's} = \frac{e^x - e^{-x}}{2}$$

$$\text{EGF for C's} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\text{required EGF} = \left(\frac{e^x - e^{-x}}{2}\right)^2 e^x = \frac{e^{3x} - 2e^x + e^{-x}}{4}$$

Example

How many sequences of n letters can be formed from A, B, and C such that the number of A's is odd and the number of B's is odd ?

$$\text{required EGF} = \frac{e^{3x} - 2e^x + e^{-x}}{4}$$

$$\text{coefficient of } x^n = \frac{1}{n!}\left(\frac{3^n - 2 + (-1)^n}{4}\right)$$

$$\text{required number} = \frac{3^n - 2 + (-1)^n}{4}$$

Example

How many ways can n people be arranged into pairs, (the pairs are not numbered) ?

$A(x)$: exponential generating function for a single pair

$$a_2 = 1, \ a_i = 0 \text{ for i} \neq 2$$

$$A(x) = \frac{x^2}{2}$$

$E(x)$: exponential generating function for arranging n people into pairs

$$E(x) = e^{\frac{x^2}{2}}$$

$$x^n \text{ term} = \frac{e_n}{n!} x^n = \left(\frac{x^2}{2}\right)^{n/2} \frac{1}{(n/2)!} \qquad e_n = \frac{n!}{2^{n/2}(n/2)!}$$

Derangements (or Hatcheck lady revisited)

d_n : number of permutations on n objects without a fixed point

$D(x)$: exponential generating function for number of derangements

A permutation on $[n]$ can be constructed by picking a subset K of $[n]$, constructing a derangement of K and fixing the elements of $[n]$-K.

Every permutation of [n] arises exactly once this way.

EGF for all permutations $= \sum_{n=0}^{\infty} \frac{n!}{n!} x^n = \frac{1}{1-x}$

EGF for permutations with all elements fixed $= \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$

$$\frac{1}{1-x} = D(x) \cdot e^x$$

Derangements

$$\frac{1}{1-x} = D(x) \cdot e^x$$

$$D(x) = e^{-x} \frac{1}{1-x}$$

$$= \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \right) \left(\sum_{n=0}^{\infty} x^n \right)$$

$$\frac{d_n}{n!} = \text{coefficient of } x^n = \left(\sum_{k=0}^{n} \frac{(-1)^k}{k!} \right)$$

$$d_n = n! \left(\sum_{k=0}^{n} \frac{(-1)^k}{k!} \right)$$