Exponential Generating Functions

\((a_0, a_1, a_2, \ldots)\): sequence of real numbers

Exponential Generating function of this sequence is the power series

\[
a(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} x^i
\]

Ordinary Generating Functions

\((a_0, a_1, a_2, \ldots)\): sequence of real numbers

Ordinary Generating Function of this sequence is the power series

\[
a(x) = \sum_{i=0}^{\infty} a_i x^i
\]

Exponential generating function examples

What is the generating function for the sequence \((1,1,1,1,\ldots)\)?

\[
\sum_{i=0}^{\infty} \frac{1}{i!} x^i = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x
\]

What is the generating function for the sequence \((1,2,4,8,\ldots)\)?

\[
\sum_{i=0}^{\infty} \frac{2^i}{i!} x^i = 1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots = e^{2x}
\]
Operations on exponential generating functions

- Addition
  \((a_n + b_n, a_i + b_i, \ldots)\) has generating function \(a(x) + b(x)\)

- Multiplication by fixed real number
  \((\alpha a_n, \alpha a_i, \ldots)\) has generating function \(\alpha a(x)\)

- Shifting the sequence to the right
  \((0, 0, 0, a_0, a_1, \ldots)\) has generating function \(x^n a(x)\)

- Shifting to the left
  \((a_k, a_{k+1}, \ldots)\) has generating function \(\sum_{i=0}^{k} a_i \cdot x^i\)

- Substituting \(\alpha x\) for \(x\)
  \((a_n, \alpha a_n, \alpha^2 a_n, \ldots)\) has generating function \(a(\alpha x)\)

- Substitute \(x^n\) for \(x\)
  \((a_0, 0, 0, a_0, 0, 0, a_0, \ldots)\) has generating function \(a(x^n)\)

- Differentiation
  \((a_n, 2a_2, 3a_3, \ldots)\) has generating function \(\frac{d}{dx}a(x)\) or \(a'(x)\)

- Integration
  \((0, a_0 + \frac{1}{2} a_1 + \frac{1}{6} a_2, \ldots)\) has generating function \(\int_0^x f(t) dt\)

- Multiplication of generating functions
  \((\sum_{n=0}^{\infty} a_n \cdot x^n)(\sum_{n=0}^{\infty} b_n \cdot x^n) = (\sum_{n=0}^{\infty} c_n \cdot x^n)\)
  \(c_n = \sum_{k=0}^{n} a_k \cdot b_{n-k}\)

Differentiation

\(a(x)\) is the exponential generating function for \((a_0, a_1, a_2, \ldots)\)

\(a(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot x^i\)

\(\frac{d}{dx}a(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot (i-1)! \cdot x^{i-1}\)

\(\frac{d}{dx}a(x)\) is the exponential generating function for \((a_1, a_2, a_3, \ldots)\)

Differentiation is equivalent to shifting the sequence to the left
Integration

\[ a(x) \] is the exponential generating function for \((a_0, a_1, a_2, \ldots)\)

\[ a(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} x^i \]

\[ \int_0^x a(t) \, dt = \sum_{i=0}^{\infty} \frac{a_i}{i!} \int_0^x t^i \, dt = \sum_{i=0}^{\infty} \frac{a_i}{i!} \frac{x^{i+1}}{(i+1)} = \sum_{i=0}^{\infty} \frac{a_i}{(i+1)!} x^{i+1} \]

\[ \int_0^x a(t) \, dt \] is the exponential generating function for \((0, a_0, a_1, a_2, \ldots)\)

Integration is equivalent to shifting the sequence to the right.

Implications of product rule

\[ C(x) = A(x)B(x) \]

Ordinary

\[ c_n = \sum_{i=0}^{n} a_i \cdot b_{n-i} \]

Useful for counting with indistinguishable objects

Exponential

\[ c_n = \sum_{k=0}^{n} \binom{n}{k} a_k \cdot b_{n-k} \]

Useful for counting with ordered objects

Interpretation of Multiplication: Product Rule

Given arrangements of type A and type B, define arrangements of type C for \(n\) labeled objects as follows:

Divide the group of \(n\) labeled objects into two groups, the First group and the Second group;
arrange the First group by an arrangement of type A and the Second group by an arrangement of type B.

\[ a_n : \text{number of arrangements of type A for } n \text{ objects} \]

\[ b_n : \text{number of arrangements of type B for } n \text{ objects} \]

\[ c_n : \text{number of arrangements of type C for } n \text{ objects} \]
Interpretation of Multiplication: Product Rule

\[ a_n : \text{number of arrangements of type A for } n \text{ people} \]
\[ b_n : \text{number of arrangements of type B for } n \text{ people} \]
\[ c_n : \text{number of arrangements of type C for } n \text{ people} \]

\[ c_n = \sum_{k=0}^{n} \binom{n}{k} a_k \cdot b_{n-k} \]

\[ a_n : \text{exponential generating function } A(x) \]
\[ b_n : \text{exponential generating function } B(x) \]
\[ c_n : \text{exponential generating function } C(x) \]

\[ C(x) = A(x)B(x) \]

\[ D_k(x) : \text{exponential generating function for arrangements of type D with exactly } k \text{ groups} \]

\[ D_k(x) = A(x)^k \]

\[ D(x) = \sum_{k=0}^{\infty} D_k(x) \]

\[ D(x) = \sum_{k=0}^{\infty} A(x)^k \]

\[ = \frac{1}{1 - A(x)} \]

\[ A(x) : \text{exponential generating function for arrangements of type A with exactly } 0 \text{ empty groups allowed} \]

Define arrangements of type D for n labeled objects as follows:

Divide the group of n labeled objects into k groups, the First group, Second group, ..., kth group \((k = 0, 1, 2, \ldots)\) and arrange each group by an arrangement of type A.

\[ D(x) : \text{exponential generating function for arrangements of type D} \]

\[ E(x) : \text{exponential generating function for arrangements of type E} \]
Example

How many sequences of \( n \) letters can be formed from A, B, and C such that the number of A’s is odd and the number of B’s is odd?

Required EGF: \( e^{3x} - 2e^x + e^{-x} \)

Coefficient of \( x^n \) = \( \frac{1}{n!} \left( \frac{3^n - 2 + (-1)^n}{4} \right) \)

Required number = \( \frac{3^n - 2 + (-1)^n}{4} \)

Example

How many sequences of \( n \) letters can be formed from A, B, and C such that the number of A’s is odd and the number of B’s is odd?

EGF for A's = \( \sum_{n \text{ odd}} \frac{x^n}{n!} = \frac{e^x - e^{-x}}{2} \)

EGF for B's = \( \frac{e^x - e^{-x}}{2} \)

EGF for C's = \( \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \)

Required EGF = \( \left( \frac{e^x - e^{-x}}{2} \right)^3 e^x = \frac{e^{3x} - 2e^x + e^{-x}}{4} \)

Example

How many ways can \( n \) people be arranged into pairs, (the pairs are not numbered)?

\( A(x) \): exponential generating function for a single pair

\( a_2 = 1, \ a_i = 0 \) for \( i \neq 2 \)

\( A(x) = \frac{x^2}{2} \)

\( E(x) \): exponential generating function for arranging \( n \) people into pairs

\( E(x) = \frac{x^2}{2} \)

\( x^n \) term = \( \frac{e^n}{n!} \cdot \frac{1}{(n/2)!} \cdot \frac{n!}{2^{n/2} (n/2)!} \)
Derangements (or Hatcheck lady revisited)

$d_n$ : number of permutations on $n$ objects without a fixed point

$D(x)$ : exponential generating function for number of derangements

A permutation on $[n]$ can be constructed by picking
a subset $K$ of $[n]$, constructing a derangement of $K$ and
fixing the elements of $[n]-K$.

Every permutation of $[n]$ arises exactly once this way.

EGF for all permutations $\sum_{n=0}^{\infty} \frac{n!}{n!} x^n = \frac{1}{1-x}$

EGF for permutations with all elements fixed $\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$

$\frac{1}{1-x} = D(x) \cdot e^x$

Derangements

$\frac{1}{1-x} = D(x) \cdot e^x$

$D(x) = e^{-x} \cdot \frac{1}{1-x}$

$= \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \right) \left( \sum_{n=0}^{\infty} x^n \right)$

$= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{(-1)^k}{k!} \right) x^n$

$\frac{d_n}{n!} = \text{coefficient of } x^n = \left( \sum_{k=0}^{n} \frac{(-1)^k}{k!} \right)$

$d_n = n! \left( \sum_{k=0}^{n} \frac{(-1)^k}{k!} \right)$